Physics 721: Homework # 4
Due in class on Wednesday Nov. 11
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Office Hours in PSC 2136: 5pm-6:30pm on Mon Nov. 2 and Mon Nov. 9

1. Bragg scattering

In this problem we will study how optical fields can be used to coherently transfer momentum to atoms.

Consider a two-level atom under the influence of two laser fields of the same polarization, each characterized by a detuning, Rabi frequency, and wavevector, \( \delta_i, \Omega_i, \mathbf{k}_i \). We will assume in the following that \( \Omega_1 = \Omega_2 \). In the proper rotating frame, the effective Hamiltonian is

\[
H = \frac{\hat{p}^2}{2m} - \hbar \Delta \langle 2 \vert \langle 2 \vert - \hbar \Omega \left[ (e^{i \mathbf{k}_1 \cdot \mathbf{r} - i \delta_1 t} + e^{i \mathbf{k}_2 \cdot \mathbf{r} + i \delta_2 t}) \vert 2 \rangle \langle 1 \vert + h.c. \right].
\]  

(1)

Here we have explicitly included the spatial degrees of freedom, and \( \Delta = (\delta_1 + \delta_2)/2, \delta = (\delta_1 - \delta_2)/2 \). We will work in a regime where \( |\Delta| \gg \Omega \) and the atom starts in the ground state with zero momentum, \( |1, \mathbf{p} = 0 \rangle \).

In this problem we ignore spontaneous emission.

a. (2 pts) Using the atomic wavefunction of the form

\[
\ket{\psi(t)} = \sum_{i=1,2} \int d\mathbf{p} \; c_{i,\mathbf{p}}(t) \ket{i, \mathbf{p}},
\]

derive the equations of motion for the probability amplitudes \( c_{1,\mathbf{p}} \) and \( c_{2,\mathbf{p}} \).

b. (2 pts.) Recalling that \( |\Delta| \gg \Omega \), use adiabatic elimination of state \( \ket{2} \) to derive the effective dynamics of the atom in the ground state. Derive an effective Hamiltonian that corresponds to these reduced equations of motion. Hint: Defining \( \mathbf{q} = \mathbf{k}_1 - \mathbf{k}_2 \) and \( \omega_p = p^2/2\hbar m \), one can go to a rotating frame \( \tilde{c}_{1,\mathbf{p}} = e^{2i\delta(t)\mathbf{q}^2/(\hbar|\mathbf{q}|^2)}c_{1,\mathbf{p}} \) to arrive at a time-independent effective Hamiltonian given by

\[
H_{\text{eff}} = \int d\mathbf{p} \; \mathcal{E}_{\mathbf{p}} \langle 1, \mathbf{p} \rangle \langle 1, \mathbf{p} \rangle - (\hbar \Omega(\mathbf{p}) \langle 1, \mathbf{p} \rangle + h \mathbf{q}) + h \Omega(\mathbf{p} - h \mathbf{q}) \langle 1, \mathbf{p} \rangle \langle 1, \mathbf{p} - h \mathbf{q} \rangle,
\]

where

\[
\mathcal{E}_{\mathbf{p}} = \frac{p^2}{2m} - \frac{\hbar \Omega^2}{\omega_{p+h\mathbf{k}_1 - \Delta}} - \frac{\hbar \Omega^2}{\omega_{p+h\mathbf{k}_2 - \Delta}} - 2 \delta \mathbf{p} \cdot \mathbf{q} / |\mathbf{q}|^2, \]

(4)

\[
\Omega(\mathbf{p}) = \frac{\hbar \Omega^2}{\omega_{p+h\mathbf{k}_1 - \Delta}}.
\]

(5)

c. (3 pts.) Find the Bragg resonance condition for a single momentum kick, i.e., for a resonant transition \( |1, 0 \rangle \rightarrow |1, h \mathbf{q} = h(\mathbf{k}_1 - \mathbf{k}_2) \rangle \), and provide a physical interpretation of your result. Under which conditions can transitions into higher momentum states be disregarded? For the conditions under which higher-order transitions can be neglected, you can leave your expressions in terms of \( \mathcal{E}_{\mathbf{p}} \) and \( \Omega(\mathbf{p}) \).

Note: Such Bragg scattering is an effective probe of excitation energy and particle correlation functions, e.g., in BEC’s (see D. Stamper-Kurn, W. Ketterle, arXiv:cond-mat/0005001).

d. (3 pts.) Find the conditions on \( \delta \) and \( \Omega \) necessary to suppress the single momentum kick \( \mathbf{p} \rightarrow \mathbf{p} + \mathbf{q} \) and enhance the double momentum kick \( \mathbf{p} \rightarrow \mathbf{p} + 2\mathbf{q} \). Consider only the case with \( \mathbf{p}_{\text{initial}} = 0 \). For this part you can leave your expressions in terms of \( \mathcal{E}_{\mathbf{p}} \) and \( \Omega(\mathbf{p}) \). Notice, however, that the quadratic dependence of \( \mathcal{E}_{\mathbf{p}} \) on \( p \) makes it much easier to satisfy this condition than if the dependence was linear. Hint: for parts c. and d., it might help to think of an analogy between the system described by \( H_{\text{eff}} \) and a multi-level atom with multiple applied external fields.
2. Optical molasses

Assume that you have a three-dimensional optical molasses in a spherical volume of radius \( r = 2 \text{mm} \). For example, one could have a pair of negatively-detuned counter-propagating beams along each of the \( \hat{x}, \hat{y} \), and \( \hat{z} \) axes. The molasses is surrounded by a gas of Rb atoms with pressure \( P = 10^{-8} \text{ Torr} \) and a temperature \( T = 300 \text{ K} \). Recall that the force law for a travelling wave on a stationary atom is given by

\[
F = \hbar k \frac{\Gamma}{2} \frac{s_0}{1 + s_0 + (2\delta/\Gamma)^2},
\]

where \( s_0 = 2\Omega^2/\Gamma^2 \) is the saturation parameter at resonance, and we will assume that the atoms are radiatively broadened (\( \gamma_{12} = \Gamma/2 \)). Here \( \Gamma \) is the spontaneous emission rate.

\( \text{a.} \) (3 pts.) Examining the motion along one dimension, estimate the maximum velocity for an atom that still allows it to be captured by the molasses. Assume that all of the laser beams have the same intensity (\( s_0 \) will continue to refer to the saturation parameter of a single laser). This capture velocity is defined as the velocity \( v_c \) of an atom entering the cloud such that it has zero velocity at the opposite edge of the cloud. In particular, in the limit of large \( v_c \), show that

\[
v_c \propto \left( -\frac{r\hbar s_0 \delta \Gamma^3}{m k^2} \right)^{1/5}. \tag{7}
\]

Hint: note the identity

\[
F_{\text{total}} = m a = m \frac{dv}{dt} = m \frac{dx}{dt} \frac{dv}{dx} = mv \frac{dv}{dx}. \tag{8}
\]

One simple way to estimate the capture velocity is to integrate this expression,

\[
\int_{v_c}^0 \frac{mv \, dv}{F_+ + F_-} = \int_{-r}^{r} dx,
\]

where \( F_{\pm} \) are the velocity-dependent forces due to the forward- and backward-propagating beams along \( x \). With a little algebra the integrand on the left-hand side can be converted into a polynomial, which can be integrated exactly. To derive Eq. (7), use the exact expressions for \( F_{\pm} \) instead of expanding them to first order in \( v \) (expanding the result and keeping the first-order term slightly overestimates the capture velocity).

\( \text{b.} \) (2 pts.) For small \( v \), show that the force on an atom along any axis satisfies

\[
F = -\alpha v, \tag{10}
\]

and find the damping coefficient \( \alpha \).

\( \text{c.} \) (3 pts.) Find the diffusion coefficient \( D \) of an atom in the three-dimensional molasses due to the absorption and spontaneous scattering of photons. Using this result and the damping coefficient \( \alpha \) obtained in part \( \text{b.} \), find the values of \( \delta \) and \( s_0 \) that maximize the diffusion time in the molasses. The diffusion time \( t_d \) is defined as the time it takes, on average, for one atom, starting at the center with zero velocity, to reach the edge of the molasses. You should assume that the diffusion time \( t_d \) is much longer than the damping time \( (\alpha/m)^{-1} \equiv \eta^{-1} \). Hint: there are several ways to do this. For example, from \( D \) and \( \alpha \) you can calculate the equilibrium temperature and thermal velocity \( v_{th} \) of atoms in the molasses. You can then model the motion of the atom as a random walk, where in the damping time the atom travels \( l = v_{th}/\eta \) before it randomly changes direction, and changes direction a total of \( \eta t_d \) times.

\( \text{d.} \) (2 pts.) Given the capture velocity from part \( \text{a.} \), we know the distribution of velocities from atoms in the surrounding gas that will be captured by the molasses. Similarly, the diffusion time gives a rate at which atoms leave the molasses. Combine this information to find the equilibrium density \( n \) of the confined gas, ignoring atom-atom interactions. How much larger is the phase space density of the cold gas compared to that of the surrounding vapor? The phase space density is given by \( \rho = n(\lambda_{DB}/2\pi)^3 \), where \( \lambda_{DB} = 2\pi\hbar/\sqrt{mk_B T} \) is the de Broglie wavelength. The spontaneous emission rate of Rb is \( \Gamma = 36 \text{ MHz} \).

Note: for more information on optical molasses see, e.g., W. Phillips’ summer school notes on Laser cooling and trapping of neutral atoms, provided on the course webpage.
3. Magneto-Optical Traps

In class we considered the force on atoms in the weak field limit at large detuning, and used those results to understand optical molasses systems. With the addition of a magnetic field gradient, \( \mathbf{B} = b_0(x\hat{x} + y\hat{y} + z\hat{z}) \), the system can form a magneto-optical trap (MOT). Here, we work explicitly with a \( J = 0 \) to \( J = 1 \) transition, as shown in the figure.

![Figure 1: a. Level diagram. b. Schematic of the one-dimensional MOT; beams of \( \sigma_{\pm} \) polarization push opposite the magnetic field gradient. The field gradient results in position-dependent Zeeman shifts of states \( |1, \pm 1\rangle \).](image)

**a. (5 pts.)** Here we will consider a one-dimensional MOT along the \( z \)-axis, as shown in the figure. Assume that the laser detuning \( \Delta \) from the center frequency of the atomic resonance is much larger than the Doppler shift and Zeeman shift. Calculate the effective detuning \( \delta(v, z) \) an atom at position \( z \) and velocity \( v \) sees due to each of the laser beams. From this expression, derive an approximate force law for a one-dimensional MOT:

\[
F/m = -\gamma v - \omega^2 z. \tag{11}
\]

Find the dependence of \( \gamma \) and \( \omega \) on the laser intensity, detuning, and magnetic field gradient \( b_0 \).

**b. (5 pts.)** Now we take typical values for a saturation parameter \( s_0 \approx 1 \), \( \gamma^{-1} = 100 \mu s \), and \( \omega = 2\pi \times 1 \) kHz. Suppose that we are trapping Na atoms \( (m = 23 \) amu, \( \lambda = 589 \) nm) with a laser intensity such that the scattering rate is \( 10^7 \) s\(^{-1}\). Determine the equilibrium 1-D temperature using these values. Estimate the size of the cloud at this temperature. Hint: there are many independent ways to calculate the equilibrium temperature. One way, for example, is to calculate the cooling rate \( dE_{\text{cool}}/dt \) one would expect from the force law above, and equate it with the heating rate \( dE_{\text{heat}}/dt \) one would expect from spontaneous emission.

Note: for more information on MOT’s see, *e.g.*, Metcalf & Straten, *Laser Cooling and Trapping*, pp. 154-162.

**c. (3 pts.)** EXTRA CREDIT: In lecture it was discussed that the “radiation trapping” force can present a limitation on the effectiveness of a MOT at sufficiently high densities. This force is due to atoms re-absorbing photons scattered from other atoms in the trap. Show that this force obeys the equation

\[
\nabla \cdot \mathbf{F}_R \propto \frac{\sigma_L \sigma_R n I}{c}, \tag{12}
\]

where \( \sigma_L \) is absorption cross-section of the laser beam, \( \sigma_R \) is the absorption cross-section of the scattered light, \( n \) is the density of the cloud, and \( I \) is the intensity of one of the trapping lasers. Hint: it may be useful to find the magnitude of the force between two atoms separated by a distance \( d \) due to the radiation of a spontaneously emitted photon from one atom and its re-absorption by a second atom. This gives an inverse square law, and application of Gauss’ Law gives the desired result. Assume that photons are only scattered twice.
4. The effect of dephasing on Doppler cooling

In this problem we will consider the effect that dephasing of the atomic coherence has on laser cooling. Recall that the evolution of the coherence operator $\sigma_{12} = \left|1\right\rangle\left\langle2\right|$ satisfies the Langevin equation

$$\frac{d}{dt}\sigma_{12} = -\gamma_{12}\sigma_{12} + \hat{F}_{12} + \text{Ham. terms.} \hspace{1cm} (13)$$

In the presence of dephasing, $\gamma_{12} > \Gamma/2$, where $\Gamma$ is the spontaneous emission rate. Here $\hat{F}_{12}$ is the noise operator acting on $\sigma_{12}$.

a. (2 pts.) Using the generalized Einstein relation, calculate the diffusion coefficients $D_{12,21}$, $D_{21,12}$, and $D_{21,21}$. In particular, show that

$$2D_{12,21} = \Gamma + (2\gamma_{12} - \Gamma) \langle\sigma_{11}\rangle, \hspace{1cm} (14)$$

$$2D_{21,12} = 2\gamma_{12} - 2D_{12,21}, \hspace{1cm} (15)$$

while $D_{12,12} = D_{21,21} = 0$. Here you can ignore the Hamiltonian terms in the evolution.

b. (3 pts.) Now consider a plane wave of constant intensity acting on the atom, with a corresponding Hamiltonian (in the rotating frame)

$$H = -\hbar\delta|2\rangle\langle2| - \hbar\left(\Omega e^{ikz}|2\rangle\langle1| + h.c.\right). \hspace{1cm} (16)$$

Including the Hamiltonian dynamics into Eq. (13), adiabatically eliminate $\sigma_{12}$ and show that in the weak field limit, the force operator $\hat{F}$ acting on the atom along $z$ is given by

$$\hat{F} = (\hbar k) \left[\frac{2\gamma_{12}|\Omega|^2}{\gamma_{12}^2 + \delta^2} + i \left(\frac{\hat{F}_{21}\Omega e^{ikz}}{\gamma_{12} + i\delta} - \frac{\hat{F}_{12}\Omega^* e^{-ikz}}{\gamma_{12} - i\delta}\right)\right], \hspace{1cm} (17)$$

where $\hat{F}_{12} = \hat{F}_{21}^\dagger$. Note that $\hat{F}$ consists of a constant mean force plus a fluctuating term $\delta\hat{F}(t)$.

c. (2 pts.) Using the operator diffusion coefficients calculated in part a., calculate the two-time expectation value $\langle\delta\hat{F}(t)\delta\hat{F}(t')\rangle = 2D\delta(t - t')$. In particular, show that the momentum diffusion coefficient $D$ is given by

$$D = (\hbar k)^2 \frac{\gamma_{12}|\Omega|^2}{\gamma_{12}^2 + \delta^2} \hspace{1cm} (18)$$

d. (3 pts.) Following the derivation presented in class, derive the minimum temperature achievable by Doppler cooling in the presence of dephasing. What does this imply, e.g., for Doppler cooling with broadband lasers?

Note: a good derivation of the Einstein relations can be found in Meystre and Sargent, pp. 330-331.
5. EXTRA CREDIT: Sideband cooling of a trapped ion

In this problem we will explore how external fields can be used to cool trapped ions into their vibrational ground states. This physics can also be generally applied to the cooling of confined neutral atoms or even of macroscopic objects. For example, the same ideas and equations can be used to describe recent experiments in which macroscopic mechanical resonators have been cooled to their motional ground state (see e.g., Teufel et al. Nature 475 p. 359-363 (2011) and Chan et al. Nature 478 p. 89-92 (2011)).

Consider a two-level atom with ground and excited states $|1\rangle$ and $|2\rangle$, respectively, sitting in a one-dimensional trapping potential whose states consist of the modes of a harmonic oscillator along $\hat{z}$. The oscillator eigenstates $|n\rangle$ have energies given by $\hbar\omega_n = \hbar n \omega_T$, where $\omega_T$ characterizes the oscillation frequency of the atom in the trap. The creation and annihilation operators of the harmonic oscillator are given by $b^{\dagger}, b$. Suppose also that an external field of Rabi frequency $\Omega$ and wavevector $k$ along the $z$-axis is applied to the system. The Hamiltonian of the system in the rotating frame is given by

$$H = -\hbar \delta |2\rangle \langle 2| + \hbar \omega_T b^{\dagger} b - \hbar \left( \Omega e^{ik\hat{z}} |2\rangle \langle 1| + h.c. \right).$$

(19)

Note that $\hat{z}$ is an operator corresponding to the position of the atom. In the following you can assume that $\Omega$ is real.

a. (2 pts.) Taking matrix elements of $b^{\dagger} b$ and $e^{ik\hat{z}}$ with respect to the oscillator states $|n\rangle$, show that in the limit of tight confinement the effective Hamiltonian for the system is given by

$$H_{LD} = \sum_{n=0}^{\infty} (-\hbar \delta + n\hbar \omega_T) |2, n\rangle \langle 2, n| + n\hbar \omega_T |1, n\rangle \langle 1, n|$$

$$-\hbar \left( \Omega |2, n\rangle \langle 1, n| + i\eta \Omega \sqrt{n+1}|2, n\rangle \langle 1, n+1| + i\eta \Omega \sqrt{n}|2, n\rangle \langle 1, n-1| + h.c. \right).$$

(20)

Find $\eta$ in terms of the mass $m$ of the atom and $\omega_T$. $\eta$ is commonly known as the “Lamb-Dicke parameter” and characterizes the degree of confinement of the atom. The limit $\eta \ll 1$ is often called the Lamb-Dicke limit. Note that different motional states are coupled via effective Rabi frequencies $\Omega_{2,n;1,n'}$ for $n' = n-1, n, n+1$. Hint: for tight confinement you can expand $e^{ik\hat{z}} \approx 1 + ik\hat{z}$.

b. (2 pts.) The equation of motion for the density matrix in the Lamb-Dicke limit is given by

$$i\hbar \dot{\rho} = [H_{LD}, \rho] + L[\rho],$$

(21)

where $L[\rho]$ is given by

$$L[\rho] = -i\hbar \sum_{n,n'} \frac{\Gamma_{n,n'}}{2} (|2, n\rangle \langle 2, n| \rho + \rho |2, n\rangle \langle 2, n| - 2|1, n'\rangle \langle 2, n| \rho |2, n\rangle \langle 1, n'|).$$

(22)

Here $\Gamma_{n,n'}$ gives the rate of spontaneous emission from $|2, n\rangle$ to $|1, n'\rangle$, and is given in the Lamb-Dicke limit by

$$\Gamma_{n,n'} = \begin{cases} \Gamma & n = n' \\ \eta^2 \alpha n \Gamma & n = n' + 1 \\ \eta^2 \alpha (n + 1) \Gamma & n = n' - 1 \\ 0 & \text{otherwise} \end{cases}.$$  

(23)

Here, $\Gamma$ is the free-space spontaneous emission rate and $\alpha$ depends on the geometry of the problem. For example, $\alpha = 2/5$ for a dipole matrix element perpendicular to $\hat{z}$ (see part e.), or $\alpha = 1/3$ for an isotropic oscillator.

Derive a closed set of equations of motion for the density matrix elements $P_{2,n} = \rho_{2,n;2,n}$, $P_{1,n} = \rho_{1,n;1,n}$, and $P_{2,n;1,n'}$. You can assume that at $t = 0$ the system starts out in an incoherent mixture of states $|1, n\rangle$, and that the field $\Omega$ is weak, to toss out any terms of order $\Omega^3$ and higher. Adiabatically eliminate the coherences to derive the following rate equations (in the weak field limit):

$$\dot{P}_{2,n} = \sum_{n'} R_{2,n;1,n'} P_{1,n'} - \sum_{n'} \Gamma_{n,n'} P_{2,n},$$

(24)

$$\dot{P}_{1,n} = -\sum_{n'} R_{2,n';1,n} P_{1,n} + \sum_{n'} \Gamma_{n',n} P_{2,n'},$$

(25)
where the pumping rates are given by

\[ R_{2,n:1,n'} = \frac{(\sum_{n''} \Gamma_{n,n''}) |\Omega_{2,n:1,n'}|^2}{(\sum_{n''} \Gamma_{n,n''}/2)^2 + (\delta - (n - n')\omega_T)^2}. \]  

(26)

c. (3 pts.) Expand the rate equations obtained in part b. to order \( \eta^2 \) and \( \Omega^2 \), and find the steady state solutions to these approximate rate equations. In particular, show that

\[ (n + 1)A_\pm P_{1,n+1} + nA_- P_{1,n-1} = (n + 1)A_- P_{1,n} + nA_+ P_{1,n}, \]

(27)

where

\[ A_\pm = \alpha R(\delta) + R(\delta \pm \omega_T), \]

(28)

\[ R(\delta) = \frac{\Gamma |\Omega|^2}{(\Gamma/2)^2 + \delta^2}. \]

(29)

Hint: note that \( R_{2,n:1,n} \propto |\Omega_{2,n:1,n}|^2 \propto \eta^2 \), and thus \( R_{2,n:1,n} \) is already accurate to order \( \eta^2 \) by setting \( \sum_{n''} \Gamma_{n:1,n''} = \Gamma \).

d. (3 pts.) Starting from the \( n = 0 \) case, show by induction that

\[ P_{1,n+1}/P_{1,n} = A_- / A_+. \]

(30)

Derive an expression for the average occupation number \( \langle n \rangle \), defined by

\[ \langle n \rangle = \sum_{n=0}^{\infty} n P_{1,n} \sum_{n=0}^{\infty} P_{1,n}. \]

(31)

Provide a physical explanation for why a steady-state solution exists only for \( \delta < 0 \). Show that for \( \delta = -\omega_T \), the final population \( \langle n \rangle \propto (\Gamma/\omega_T)^2 \) in the limit of large \( \omega_T \). In other words, by making the trap sufficiently tight we can make the average occupation number \( \langle n \rangle \ll 1 \), so the atom will be in the ground state with very good probability.

e. (3 pts.) EXTRA EXTRA CREDIT: In the Lamb-Dicke limit, derive the result

\[ \Gamma_{n,n-1} = \frac{2}{5} \eta^2 n \Gamma, \]

(32)

\[ \Gamma_{n,n+1} = \frac{2}{5} \eta^2 (n + 1) \Gamma, \]

(33)

for an atom whose dipole matrix element \( \langle 2|d|1 \rangle \) is perpendicular to \( \hat{z} \).