Understanding cold atomic and molecular collisions

1. Basics

Paul S. Julienne

Joint Quantum Institute
NIST and The University of Maryland

Thanks to many colleagues in theory and experiment who have contributed to this work

http://www.jqi.umd.edu/

Supported by an AFOSR MURI
Cold Collisions

“Good” -- Essential interactions for control and measurement
“Bad” -- Source of trapped atom loss, heating, and decoherence

- Atom-atom collisions can be quantitatively understood and controlled--essential for quantum gas studies.
- Basic concepts, illustrated by examples.
- Part 1: What is a scattering length, and why is it significant? Why is the long range potential so important?
- Part 2: Magnetically tunable “Feshbach” resonances are a key to measurement and control.
- Part 3(?): What is special about molecules? Molecules are different from atoms. What about few-body phenomena?
Some resources

PSJ, Ch. 6 of *Cold Molecules*, ed. by R. Krems et al.
also arXiv:0902.1727, threshold bound and scattering states

Chin, Grimm, PSJ, Tiesinga, Rev. Mod. Phys. 82, 1225 (2010)
review of Feshbach resonances

PSJ, Faraday Discussions 142, 361 (2009)
ultracold molecules: a case study with KRb

Kohler, Goral, PSJ, Rev. Mod. Phys. 78, 1311 (2006)
review of molecule formation through Feshbach res.

review of cold atom photoassociation

quantum defect theory for cold collisions
Two kinds of collision

Elastic--do not change internal state $a + b \rightarrow a + b$

- Thermalization
- Evaporation
- Mean field of BEC
- BEC-BCS crossover in Fermi gases
- Equation of state
- Phase change--quantum logic gates

Inelastic--change internal state $a + b \rightarrow a' + b' + \Delta E$

- Spin relaxation
- Loss of trapped atoms, gas lifetime
- Decoherence
- Spinor condensates ($\Delta E=0$)
- Three-body, detect resonances
- Photoassociation
Spin degeneracy
Total angular momentum

Born-Oppenheimer approximation = potential energy curve
Li $1s^2 2s^2 2S_{1/2}$ atom

Potentials like $H_2$ molecule

Similar for Na, K, Rb, Cs

$V(R)/k_B$ (K)

$R (a_0)$
From Quéméner and Julienne, Chemical Reviews 112, 4949 (2012) Ultracold Molecules Under Control!

\[
\lambda = \frac{\hbar}{mv} \gg 1 \mu m
\]
$\bar{E} = \frac{\hbar^2}{2\mu\bar{a}^2}$

$v_{\text{dW Energy}}$

$\bar{a} = \frac{1}{2} \frac{\Gamma(3/4)}{\Gamma(5/4)} \left( \frac{2\mu C_6}{\hbar^2} \right)^{1/4}$

$V(R) \sim -C_n/R^n$

Short-range

$1 \text{ eV}$

$10^{-4} \text{ eV}$

$10^{-10} \text{ eV (1 } \mu\text{K)}$

Long-range

Separated

$V(R)$

$0.02 - 5 \text{ eV}$

$R_{\text{bond}}$

$R$

$\bar{a}$

Fig. 2, PSJ Faraday Disc 142, 361 (2009)
Collision of two atoms

Separate center of mass $R_{CM}$ and relative $R$ motion with reduced mass $\mu$.

Expand $\Psi(R,E)$ in relative angular momentum basis $lm$.

$l = 0, 1, 2...$ $s$-, $p$-, $d$-waves, ...

Potential energy: $V(R) + \frac{\hbar\ell(\ell + 1)}{2\mu R^2}$ $\rightarrow$ phase shift $\eta_\ell(E)$

Neutral atoms (S-state): $V(R) \rightarrow -C_6/R^6$ van der Waals

Solve Schrödinger equation for bound and scattering $\Psi(R,E)$

$\rightarrow$ bound states $E_n$

$\rightarrow$ scattering phases, amplitudes
S-wave scattering phase shift

\[ \Psi(R) \rightarrow \sin(kR + \eta) \]

Wavelength \( \lambda = \frac{2\pi}{k} \)

Noninteracting atoms

Interacting atoms

Phase

\( \eta = 0 \)

\( \eta \rightarrow -ka \) as \( k \rightarrow 0 \)
The graph shows the wave function $\Psi(R)$ as a function of interatomic separation $R$ in atomic units ($a_0$), with $E/k_B = 1 \mu K$. The graph illustrates oscillations with a peak at $1 \mu m$. The $x$-axis represents the interatomic separation, and the $y$-axis represents the wave function value.
$\Psi(R)$

$R$ (atomic units $a_0$)

$k$-independent node

1 mK

1 $\mu$K

10 nm
Cross section $\sigma$

Classical balls with distance of closest approach $d$ (diameter)

Define an area with $\sigma = \pi d^2$ \(10^{-12} \text{ cm}^2\)

And a collision rate $\Gamma = n v \sigma$\ (typical: 1 s\(^{-1}\), MOT \(10^4\) s\(^{-1}\), BEC)

Rate constant $K = v \sigma$

Time between collisions $= 1/\Gamma = 1/(Kn) \approx 1\text{s} \ (\text{MOT}), 100\mu\text{s} \ (\text{BEC})$
Classical picture

No potential

$|ab\rangle \xrightarrow{\vec{R}(t)} |ab\rangle$  

$|\vec{L}| = |\vec{R} \times \vec{p}| = b\rho$

Centrifugal potential

$V(R) = \frac{L^2}{2\mu R^2}$

$E = \frac{p^2}{2\mu}$

Centrifugal barrier

With potential

$|ab\rangle \xrightarrow{\vec{R}(t; b)} |ab\rangle$

Interaction region

$E = 0$
Quantum scattering theory

\[ \left( e^{i\vec{k} \cdot \vec{R}} + f(\Omega) \frac{e^{i\vec{k}R}}{R} \right) |ab\rangle \]

atoms/s scattered flux into \( d\Omega \)

plane wave flux (atoms/cm\(^2\)/s)

\[ = \frac{d\sigma}{d\Omega} = |f(\Omega)|^2 \]

\[ \sigma_{\text{tot}} = \int d\Omega \frac{d\sigma}{d\Omega} \]
Partial wave expansion

Expansion of a plane wave (Messiah, Quantum Mechanics, Vol.1, Appendix B.III):

\[ e^{i \mathbf{k} \cdot \mathbf{R}} = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} i^{\ell} Y_{\ell m}^*(\mathbf{k}) Y_{\ell m}(\mathbf{R}) j_{\ell}(kR) \]

Geometric / Dynamic

solution to the radial Schrödinger equation for the centrifugal potential

At large $R$: \( \sin (kR - \pi \ell/2) \)

When $V(R)$ is present: \( \sin (kR - \pi \ell/2 + \eta_{\ell}(E)) \)
How do we get the S-matrix, or bound states?

Coupled channels expansion:

$$\Psi_\alpha(R, E) = \sum_{\alpha'} \frac{\phi_{\alpha', \alpha}^+(R, E)}{R} |\alpha'\rangle$$

Solve matrix Schrödinger equation

$$\mathbf{H}\Psi(R, E) = E\Psi(R, E)$$

Extract $S$ from solution at large $R >> R_{vdW}$

Potential matrix $\mathbf{V}_{\alpha\alpha'}(R)$

$$M_{tot} = M_1 + M_2 + m_\ell$$ conserved

Electronic (Born–Oppenheimer) $V(R)$ does not change $\ell$

Anisotropic (spin–dependent) potential changes $\ell$
Collision cross section

Solve Schrödinger equation for each $\ell$

Get phase shift $\eta_\ell(E)$

$$\sigma(E) = \frac{4\pi}{k^2} \sum_\ell (2\ell + 1) \sin^2 \eta_\ell(E)$$

Identical bosons: even $\ell$
Identical fermions: odd $\ell$
Nonidentical species: all $\ell$

van der Waals potential:

$$\eta_\ell(E) \rightarrow -Ak \quad s\text{-wave as } k \rightarrow 0$$
$$\eta_\ell(E) \rightarrow -(A_1k)^3 \quad p\text{-wave as } k \rightarrow 0$$
$$\eta_\ell(E) \propto k^4 \quad d\text{-wave and higher as } k \rightarrow 0$$
S-wave collision summary

If only a single s-wave channel, \( S_{\alpha\alpha} = e^{2i\eta} \rightarrow e^{-2ika} \) as \( k \rightarrow 0 \)

If inelastic channels \( \alpha' \neq \alpha \) exist, unitarity ensures

\[
|S_{\alpha\alpha}|^2 = 1 - \sum_{\alpha' \neq \alpha} |S_{\alpha\alpha'}|^2 = 1 - 4kb_{\alpha}
\]

Thus,

\[ S_{\alpha\alpha} = e^{-2ik(a-ib)} \] as \( k \rightarrow 0 \)

Complex scattering length \( a-ib \)

\[
\sigma_{\alpha\alpha} = 4\pi(a^2 + b^2)
\]

\[
K_{loss} = \sum_{\alpha' \neq \alpha} K_{\alpha\alpha'} = 2\frac{\hbar}{\mu} b
\]
Collision cross sections and rate constants

Scattering channels

Start in initial channel: \( \{ab\} \ell m = \alpha \)

Exit in final channel: \( \{a'b'\} \ell' m' = \alpha' \)

Transition amplitudes \( T_{\alpha \alpha'} = \delta_{\alpha \alpha'} - S_{\alpha \alpha'} \) are expressed in terms of the elements of the unitary S-matrix.

Elastic cross section
\[
\sigma^\text{el}_{\alpha}(E) = \frac{\pi}{k^2} \left| 1 - S_{\alpha \alpha}(E) \right|^2
\]

Inelastic cross section
\[
\sigma^\text{loss}_{\alpha}(E) = \frac{\pi}{k^2} \left( 1 - |S_{\alpha \alpha}(E)|^2 \right)
\]

\[
\sum_{\alpha' \neq \alpha} |S_{\alpha \alpha'}(E)|^2 = 1 - |S_{\alpha \alpha}(E)|^2 \text{ since } S \text{ is unitary.}
\]

Sum over all contributing \( \ell m \) for a given \( \{ab\} \) to get the TOTAL (observable) cross section.
Rate constants

Loss rate constant

\[ K_{\alpha}^{\text{loss}}(E) = \nu \sigma_{\alpha}^{\text{loss}}(E) = \frac{\hbar}{\mu} b_{\alpha}(E) \]

where \( b_{\alpha}(E) = \frac{1 - |S_{\alpha\alpha}(E)|^2}{k} \) has units of length.

\[ K_{\alpha}^{\text{loss}}(E) = 2\frac{\hbar}{\mu} b_{\alpha}(E) = 4.2 \times 10^{-11} \text{ cm}^3/\text{s} \frac{b[a_0]}{\mu[\text{amu}]} \]

Typical values:

“Allowed” \( b \sim 10-100 a_0 \)

“Forbidden” \( b << 1 a_0 \)

Upper bound \( 4b = k^{-1} = \lambda/2\pi \)
Elastic cross section $\sigma$ for like Na atoms

Many (even) partial waves
Threshold properties (elastic only)

\[ \sigma(E) = \frac{4\pi}{k^2} \sin^2(kA) = 4\pi A^2 \text{ as } k \to 0 \]

(8\pi A^2 \text{ for identical bosons})

Upper bound = 1 (unitarity limit)

\[ \sigma(E) = \frac{4\pi}{k^2} \propto \frac{1}{E} \]

\[ = 4\pi \left( \frac{\lambda}{2\pi} \right)^2 \]
Atomic and molecular collision rates


\[
\frac{dn_a}{dt} = -Kn_an_b = -\frac{1}{\tau}n_a \quad \text{where} \quad \frac{1}{\tau} = Kn_b
\]

\[
K = \frac{1}{Q_T} \frac{k_BT}{h} f_D
\]

where

\[
\frac{1}{Q_T} = \Lambda_T^3 = \left(\frac{h}{2\pi\mu k_BT}\right)^{\frac{3}{2}}
\]

\(Q_T = \text{translational partition function}\)

\(\Lambda_T = \text{thermal de Broglie wavelength of pair}\)

\[
f_D = \int_0^\infty \sum_{\ell m} \left(1 - |S(\ell m)|^2\right) e^{-E/(k_BT)} dE/(k_BT)
\]

\[
\frac{1}{\tau} = Kn_b = (n\Lambda_T^3) \frac{k_BT}{h} f_D
\]

Phase Space density Time scale Dynamics

Replace

\[
1 - |S(\ell m)|^2 \quad \text{by} \quad 1 - |S(\ell m)|^2
\]

for elastic collisions
Van der Waals potentials
Van der Waals potential

Write the Schrödinger equation in length and energy units of

\[ R_{\text{vdW}} = \frac{1}{2} \left( \frac{2\mu C_6}{\hbar^2} \right)^{\frac{1}{4}} \]

or

\[ \bar{a} = \frac{\Gamma(3/4)}{\Gamma(5/4)} R_{\text{vdW}} = 0.956 R_{\text{vdW}} \]


\[ E_{\text{vdW}} = \frac{\hbar^2}{2\mu R_{\text{vdW}}^2} \]

The potential becomes

\[ -\frac{16}{r^6} + \frac{\ell(\ell + 1)}{r^2} \]

in vdw units.

This potential has exact analytic solutions and many useful properties.

and Chin et al., Feshbach review
“Size” of vdW potential

\[ \langle R \rangle = \frac{a}{2} \]

<table>
<thead>
<tr>
<th></th>
<th>( R_{vdw}(a_0) )</th>
<th>( E_{vdw}(mK) )</th>
<th>(MHz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(^6)Li</td>
<td>31</td>
<td>29</td>
<td>614</td>
</tr>
<tr>
<td>(^{40})K</td>
<td>65</td>
<td>1.0</td>
<td>21</td>
</tr>
<tr>
<td>(^{85})Rb</td>
<td>83</td>
<td>0.35</td>
<td>7.3</td>
</tr>
<tr>
<td>(^{133})Cs</td>
<td>101</td>
<td>0.13</td>
<td>2.7</td>
</tr>
</tbody>
</table>
\[ A = +101 \, a_0 \]

\[ A_s = -96 \, a_0 \]

Scattering and last bound state near threshold
(normalized to same value at small \( R \))
Bound states from van der Waals theory

\[ a = \pm \infty \]

-300
-250
-200
-150
-100
-50
0

\( s \quad p \quad d \quad f \quad g \quad h \)

Adapted from Gao, Phys. Rev. A 62, 050702 (2000); Figure from Chin et al., review
Noninteracting atoms

\[ \Psi \sim \frac{\sin(kR)}{\sqrt{k}} \]

Interacting atoms

\[ \Psi \sim \frac{\sin(k(R - A))}{\sqrt{k}} \]

Scattering lengths for Yb ground state model (in $a_0$ units)

<table>
<thead>
<tr>
<th></th>
<th>168</th>
<th>170</th>
<th>171</th>
<th>172</th>
<th>173</th>
<th>174</th>
<th>176</th>
</tr>
</thead>
<tbody>
<tr>
<td>168</td>
<td>252(6)</td>
<td>117(1)</td>
<td>89(1)</td>
<td>65(1)</td>
<td>39(1)</td>
<td>2(2)</td>
<td>-360(30)</td>
</tr>
<tr>
<td>170</td>
<td>117</td>
<td>64(1)</td>
<td>37(1)</td>
<td>-2(2)</td>
<td>-81(4)</td>
<td>-520(50)</td>
<td>209(4)</td>
</tr>
<tr>
<td>171</td>
<td>89</td>
<td>37</td>
<td>-3(2)</td>
<td>-84(5)</td>
<td>-580(60)</td>
<td>430(20)</td>
<td>142(2)</td>
</tr>
<tr>
<td>172</td>
<td>65</td>
<td>-2</td>
<td>-84</td>
<td>-600(60)</td>
<td>420(20)</td>
<td>201(3)</td>
<td>106(1)</td>
</tr>
<tr>
<td>173</td>
<td>39</td>
<td>-81</td>
<td>-580</td>
<td>420</td>
<td>199(3)</td>
<td>139(2)</td>
<td>80(1)</td>
</tr>
<tr>
<td>174</td>
<td>2</td>
<td>-520</td>
<td>430</td>
<td>201</td>
<td>139</td>
<td>105(1)</td>
<td>55(1)</td>
</tr>
<tr>
<td>176</td>
<td>-360</td>
<td>209</td>
<td>142</td>
<td>106</td>
<td>80</td>
<td>55</td>
<td>-24(2)</td>
</tr>
</tbody>
</table>
Yb a versus reduced mass $\mu$

Scattering length $a$ [Bohr]

$2\mu$ [amu]

- Mass scaled
- Calculated for specific isotopes
Number of bound states in $V(R) =$ Int \[ \left[ \frac{\Phi}{\pi} - \frac{5}{8} \right] + 1 \]

Simplest model:

\[ V(R) = -\frac{C_6}{R^6} \text{ for } R_{in} < R \leq \infty \]

\[ V(R) = \infty \text{ for } 0 < R \leq R_{in} \]
Mth level

1st energy bin

n=-1

2nd energy bin

n=-2

(N+1)th level

n=-1

n=-2

Control parameter
Last bound state energies versus mass

Solid: $J=0$
Dashed: $J=2$
\( \sigma(E) \, [\text{cm}^2] \)

\( E/k_B \, (\mu \text{K}) \)

solid: s + (p) + d

dashed: s-only

\( 170 \)

\( 172 \)

\( 173 \)

\( 174 \)

\( 176 \)

\( 171 \)
$^{174}\text{Yb}_2$
Semiclassical considerations

**WKB phase-amplitude form:**

\[ \phi_{WKB}^R(R, E) = \alpha(R, E) \sin \beta(R, E) \]

\[ \alpha(R, E) = \frac{1}{k(R, E)^{\frac{1}{2}}} \]

\[ \beta(R, E) = \int_{R_t}^{R} k(R', E) dR' + \frac{\pi}{4} \]

\[ k(R, E) = \left( \frac{2\mu}{\hbar^2}(E - V(R)) \right)^{\frac{1}{2}} = \frac{2\pi}{\lambda(R, E)} \]

**Validity criterion:**

\[ \frac{d\lambda(R, E)}{dR} \ll 1 \]
Semiclassical considerations continued

\[ \hat{f}(R, 0) = \alpha(R, 0) \sin \beta(R, 0) \quad f(R, E) \rightarrow \frac{1}{\sqrt{k}} \sin(kR + \eta(E)) \]

For \( R \ll R_{\text{vdw}} \)

\[ f(R, E) = C(E)^{-1} \hat{f}(R, E) = C(E)^{-1} \hat{f}(R, 0) \]


\[ \alpha(R, E) = \frac{1}{k(R, E)^{\frac{1}{2}}} \]

\[ \beta(R, E) = \int_{R_t}^{R} k(R', E) dR' + \frac{\pi}{4} \]
The End
Understanding cold atomic and molecular collisions

1. Feshbach resonances

Paul S. Julienne

Joint Quantum Institute
NIST and The University of Maryland

Thanks to many colleagues in theory and experiment who have contributed to this work

http://www.jqi.umd.edu/

Supported by an AFOSR MURI
Some examples of Feshbach resonances

Cesium theory


Sodium BEC

S. Inouye et al
An example for $E \rightarrow 0$


$^{85}\text{Rb BEC (below 15 nK)}$
Tunable scattering resonances used for

Making $^{40}\text{K}_2$ molecules


Making $^{133}\text{Cs}_2$ molecules

\[ ^6\text{Li } F=1/2, M=+1/2 + F=1/2, M=-1/2 \]

\[ ^87\text{Rb } F=1, M=+1 + F=1, M=+1 \]

\[ ^{14}\text{Na } F=1, M=+1 + F=1, M=+1 \]

\[ ^{133}\text{Cs } F=3, M=+3 + F=3, M=+3 \]
Molecular physics of Li+Li is well-known

Quantum degenerate Fermi mixtures
BEC-BCS crossover
Strongly interacting gas
Equation of state

Coupled channels calculations based on accurately known potentials
All spin-dependent interactions treated in Hamiltonian
2 free parameters: S and T scattering lengths
$^6\text{Li} \ a+b$ Scattering Length vs. B

Model from Bartenstein et al., PRL 94, 103201 (2005)
Data: Bartenstein et al., PRL 94, 103201 (2005)

\[ E(B) = -\frac{\hat{h}^2}{2\mu a(B)^2} \]

\[ E(B) = -\frac{\hat{h}^2}{2\mu(a - \bar{a})^2} \]

834.1(1.5) G
$B = 700 \, \text{G}$

$A(B) = 1630 \, a_0$

\[ \phi_b(R) = \sqrt{\frac{2}{a}} e^{-R/a} \]
Long history of resonance scattering

O. K. Rice, J. Chem. Phys. 1, 375 (1933)

U. Fano, Nuovo Cimento 12, 154 (1935)

J. M. Blatt and V. F. Weisskopf, Theoretical Nuclear Physics (1952)

H. Feshbach, Ann. Phys. (NY) 5, 357 (1958); 19, 287 (1962)


Separation of system into:
- An (approximate) bound state
- A scattering continuum
- with some coupling between them
Resonant Scattering Picture
(following U. Fano, Phys. Rev. 124, 1866 (1961); see arXiv:0812.1486)

Bound state

| \langle n \rangle |

\begin{align*}
\eta(E) &= \eta_{bg} + \eta_{res}(E) \\
\eta_{res} &= -\tan^{-1} \left( \frac{1}{2} \frac{\Gamma_n}{E - E_n - \delta E_n} \right) \\
\Gamma_n &= 2\pi |\langle n|V|E \rangle|^2 \\
\delta E_n &= \int \frac{|\langle n|V|E' \rangle|^2}{E_n - E'} dE'
\end{align*}
Threshold Resonant Scattering

\[ E_n = \delta \mu (B - B_n) \]

\[ E = 0 \]

\[ \eta(E, B) = \eta_{bg}(E) - \tan^{-1} \left( \frac{1}{2} \frac{\Gamma(E)}{E - E_n - \delta E_n(E)} \right) \]

As \( E \to 0 \)

\[ \eta_{bg} = -k a_{bg} \]

\[ \frac{1}{2} \Gamma_n(E) = (k a_{bg}) \delta \mu \Delta_n \]

\[ a(B) = a_{bg} \left( 1 - \frac{\Delta_n}{B - B_0} \right) \]

Shifted \( B_0 = B_n + \delta B_n \)
$^6\text{Li}~a+b$ Scattering Length vs. $B$

Model from Bartenstein et al., PRL 94, 103201 (2005)
Classification of resonances by strength

**Resonance strength**

\[ s_{\text{res}} = \frac{a_{bg}}{\bar{a}} \frac{\delta \mu \Delta}{\bar{E}} \]

See Kohler et al, Rev. Mod. Phys. 78, 1311 (2006)
And Chin, Grimm, Julienne, Tiesinga, arXiv: 0812.1486

\[ a(B) = a_{bg} \left( 1 - \frac{\Delta}{B - B_0} \right) \]

For magnetically tunable resonances:

\[ E_c = \delta \mu (B - B_c) \]

Bound state norm \( Z \) as \( E \to 0 \) and \( B \to B_0 \)

\[ Z = \zeta^{-1} \left| \frac{B - B_0}{\Delta} \right| \]

\[ \zeta = \frac{1}{2} s_{\text{res}} \frac{a_{bg}}{\bar{a}} \]
From arXiv:0812.1486
$s_{\text{res}} = 59$

$\Delta = 300 \text{ G}$

Entrance channel dominated

$\Delta = 180 \text{ G}$

Closed channel dominated

$\color{red}{6}\text{Li} \ ab$

$\color{red}{7}\text{Li} \ aa$

$E/k_B (\text{mK})$

$B (\text{Gauss})$

Color: $\sin^2 \eta(E)$
$s_{\text{res}} = 0.49$  
$s_{\text{res}} = 59$
\[ s_{\text{res}} = 59 \]
\[ \Delta = 300 \, \text{G} \]

Entrance channel dominated

\[ \begin{align*}
\text{\textsuperscript{6}Li} \, \text{ab} & \\
\text{\textsuperscript{7}Li} \, \text{aa} & \\
\end{align*} \]

Color: \( \sin^2 \eta(E) \)
The diagram illustrates the energy level of $^6{\text{Li}}$ as a function of the magnetic field. The energy is given in MHz and the magnetic field in G. The plot shows the transition between bound-bound and bound-free states at different magnetic fields. Notable points of interest are marked with arrows labeled "New", "ac", and "ab". The references for these points are:

Coupled channels fit, PSJ & J. Hutson, arxiv:1404.2623 (full Hamiltonian)

$^7\text{Li}$

$E/h$ (MHz)

B (G)

Calculated
Rice
Paris
Universal energy: 
\[ E^U = -\frac{\hbar^2}{2\mu a^2} \]

Reduced E and length: 
\[ \epsilon = \frac{E}{\bar{E}} \text{ and } r = a/\bar{a} \]

\[ \epsilon^U = -\frac{1}{r^2} \]

\[ \epsilon^U r^2 = -1 \]
\[ \epsilon_U = -\frac{1}{r^2} \]  

Universal

\[ \epsilon_{GF} = -\frac{1}{(r - 1)^2} \]  

Gribakin and Flambaum, PRA 48, 546 (1993)

\[ \epsilon_G = \epsilon_{GF} \left( 1 + \frac{g_1}{r - 1} + \frac{g_2}{(r - 1)^2} \right) \]

\[ g_1 = \Gamma \left( \frac{1}{4} \right)^4 / (6\pi^2) - 2 \approx 0.9179 \]

\[ g_2 = (5/4)g_1^2 - 2 \approx -0.9468 \]

The End