COHERENCE PROPERTIES OF A BOSE-EINSTEIN CONDENSATE

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OUTLINE

- Description of the problem
- Framework: Bogoliubov theory
- Spatial coherence
- Temporal coherence
  - $N$ fluctuates
  - $N$ fixed, $E$ fluctuates: Canonical ensemble
  - $N$ fixed, $E$ fixed: Microcanonical ensemble
DESCRIPTION OF THE PROBLEM
A single-spin state Bose gas prepared at equilibrium:

- Spatially homogeneous, periodic boundary conditions.
- Prepared with $N$ atoms, in the regime $T \ll T_c$ of an almost pure condensate.
- Interactions with a $s$-wave scattering length $a > 0$.
- Weakly interacting regime $(\rho a^3)^{1/2} \ll 1$.
- The gas is totally isolated in its evolution.

Spatial coherence of the gas:

- Determined by the measured first-order coherence function, $g_1(r) = \langle \hat{\psi}^\dagger(r)\hat{\psi}(0) \rangle$ (Esslinger, Bloch, Hänsch, 2000).
- Expected: In thermodynamic limit, $g_1$ tends to condensate density $\rho_0 > 0$ at infinity.
- This is long-range order.
Coherence time of the condensate:

- **Defined as the decay time of the measurable condensate mode coherence function**, \( \langle a_0^\dagger(t) a_0(0) \rangle \), where \( a_0 \) is the annihilation operator in mode \( k = 0 \).

- **At zero temperature**, no decay, \( \langle a_0^\dagger(t) a_0(0) \rangle \sim \langle N_0 \rangle e^{i\mu_0 t/\hbar} \), coherence time is infinite (Beliaev, 1958).

- **What happens at finite temperature** \( T > 0 \)? To our knowledge, the problem was still open in 1995.

- **One expects infinite coherence time in thermodynamic limit.**

- **For finite size**: By analogy with laser, one expects finite coherence time due to condensate phase diffusion.
FRAMEWORK: BOGOLIUBOV THEORY
Bogoliubov theory

• **Lattice model Hamiltonian:**

\[ H = \sum_r b^3 \left[ \hat{\psi}^\dagger h_0 \hat{\psi} + \frac{g_0}{2} \hat{\psi}^\dagger \hat{\psi}^\dagger \hat{\psi} \hat{\psi} \right] \]

• **Spatially homogeneous case:**

\[ h_0 = -\frac{\hbar^2}{2m} \Delta r. \]

• **Bare coupling constant**

\[ g_0^{-1} = g^{-1} - \int_{FBZ} \frac{d^3 k}{(2\pi)^3} \frac{m}{\hbar^2 k^2}, \quad g = 4\pi \hbar^2 a/m. \] Gives \[ g_0 = g/(1 - C_3 a/b). \] Here \( 0 < a \ll b. \)

• **Expansion of Hamiltonian around pure condensate:**

\[ \hat{\psi}(r) = \phi(r) \hat{a}_0 + \hat{\psi}_\perp(r) \]

with \( \phi(r) = 1/L^{3/2}. \) **Key point:** Eliminate amplitude \( \hat{a}_0 \) in condensate mode:

\[ \hat{n}_0 = \hat{N} - \hat{N}_\perp \]

with \( \hat{n}_0 = \hat{a}_0^\dagger \hat{a}_0 \) and \( \hat{N}_\perp = \sum_r b^3 \hat{\psi}_\perp^\dagger \hat{\psi}_\perp. \)
Elimination of the condensate phase

- Modulus-phase representation (Girardeau, Arnowitt, 1959):
  \[ \hat{a}_0 = e^{i\hat{\theta} \hat{n}_0^{1/2}} \]
  with hermitian operator \( \hat{\theta} \), \([\hat{n}_0, \hat{\theta}] = i\).

- Cf. position \( \hat{x} \) and momentum \( \hat{p} \) operator of a particle:
  \[
  [\hat{x}, \hat{p}] = i\hbar \implies e^{i\hat{p}a/\hbar}\langle x \rangle = \langle x - a \rangle
  \]
  \[
  [\hat{n}_0, \hat{\theta}] = i \implies e^{i\hat{\theta}}|n_0 : \phi\rangle = |n_0 - 1 : \phi\rangle
  \]
  then \( \hat{a}_0 \) has the right matrix elements.

- This fails when the condensate mode is empty:
  \[ e^{i\hat{\theta}}|0 : \phi\rangle \nRightarrow | - 1 : \phi\rangle \]

- Redefinition of non-condensed field (Castin, Dum; Gardiner, 1996); remains bosonic, but conserves \( \hat{N} \):
  \[ \hat{\Lambda}(r) = e^{-i\hat{\theta}}\hat{\Psi}_\perp(r) \]
Expansion of $H$ to second order in $\hat{\psi}_\perp$:

$$H_{\text{Bog}} = \frac{g_0 N^2}{2 L^3} + \sum_r b^3 \left[ \hat{\Lambda}^\dagger (\hbar_0 - \mu_0) \hat{\Lambda} + \mu_0 \left( \frac{1}{2} \hat{\Lambda}^2 + \frac{1}{2} \hat{\Lambda}^\dagger^2 + 2 \hat{\Lambda}^\dagger \hat{\Lambda} \right) \right]$$

Formally grand canonical for non-condensed modes, with chemical potential $\mu_0 = g_0 \rho$.

Elastic interaction $C - NC$: Hartree-Fock

$$C, 0 + NC, k \rightarrow C, 0 + NC, k$$

Inelastic interaction $C - NC$: Landau superfluidity

$$C, 0 + C, 0 \rightarrow NC, k + NC, -k$$

Not forbidden by energy conservation.
Normal form for the Hamiltonian:

- $H_{\text{Bog}}$ quadratic, hence linear equations of motion:

$$i\hbar\partial_t \begin{pmatrix} \Lambda \\ \Lambda^\dagger \end{pmatrix} = \begin{pmatrix} h_0 + \mu_0 & \mu_0 \\ -\mu_0 & -(h_0 + \mu_0) \end{pmatrix} \begin{pmatrix} \Lambda \\ \Lambda^\dagger \end{pmatrix} \equiv \mathcal{L} \begin{pmatrix} \Lambda \\ \Lambda^\dagger \end{pmatrix}$$

- Expansion on eigenmodes of eigenenergies $\pm \epsilon_k$:

$$\begin{pmatrix} \Lambda \\ \Lambda^\dagger \end{pmatrix} = \sum_{k \neq 0} \frac{e^{i k \cdot r}}{L^{d/2}} \begin{pmatrix} U_k \\ V_k \end{pmatrix} \hat{b}_k + \frac{e^{-i k \cdot r}}{L^{d/2}} \begin{pmatrix} V_k \\ U_k \end{pmatrix} \hat{b}_k^\dagger$$

because exchanging $\Lambda$ and $\Lambda^\dagger$ equivalent to time reversal.

- Bosonic commutation relations for $U_k^2 - V_k^2 = 1$:

$$U_k + V_k = \frac{1}{U_k - V_k} = \left( \frac{\hbar^2 k^2}{2m} \right) \left( \frac{2\mu_0 + \hbar^2 k^2}{2m} \right)^{1/4}$$
• A grand-canonical ideal gas of bosonic quasi-particles:

\[ H_{\text{Bog}} = E_0 + \sum_{k \neq 0} \epsilon_k \hat{b}_k^\dagger \hat{b}_k \quad \text{with} \quad \epsilon_k = \left[ \frac{\hbar^2 k^2}{2m} \left( \frac{\hbar^2 k^2}{2m} + 2\mu_0 \right) \right]^{1/2} \]

Bogoliubov spectrum
SPATIAL COHERENCE
Consistency check

In thermodynamic limit:

- **Non-condensed fraction:**
  \[
  \frac{\langle N_\perp \rangle}{N} = \frac{\langle \hat{\Lambda}^\dagger \hat{\Lambda} \rangle}{\rho} = \frac{1}{\rho} \int \frac{d^3k}{(2\pi)^3} \left[ \frac{U_k^2 + V_k^2}{e^{\beta \epsilon_k} - 1} + V_k^2 \right]
  \]

- No ultraviolet \((k \to \infty)\) divergence: \(V_k^2 = O(1/k^4)\)
- No infrared \((k \to 0)\) divergence: \(U_k^2, V_k^2 = O(1/k)\).
- Small for \(T \ll T_c\) and \((\rho a^3)^{1/2} \ll 1\).
- **First order coherence function** \(g_1(r) = \langle \hat{\psi}^\dagger(r) \hat{\psi}(0) \rangle:\)
  \[
  g_1(r) = \rho - \int \frac{d^3k}{(2\pi)^3} (1 - \cos k \cdot r) \left[ \frac{U_k^2 + V_k^2}{e^{\beta \epsilon_k} - 1} + V_k^2 \right]
  \]
  tends to the condensate density for \(r \to \infty\).
In lower dimensions:

- In 2D for $T > 0$ and in 1D $\forall T$, the non-condensed fraction has infrared divergence. No BEC in thermodynamic limit (Mermin, Wagner, 1966; Hohenberg, 1967).

- Quasi-condensate (weak density fluctuations, weak phase gradients) (Popov, 1972). One can save the idea of Bogoliubov by applying it to a modulus-phase representation of the field operator $\hat{\psi}$.

- $g_1^{Bog}(r) \to -\infty$ at infinity, but remarkably (Mora, Castin, 2003):

$$g_1^{QC}(r) = \rho \exp \left[ \frac{g_1^{Bog}(r)}{\rho} - 1 \right].$$

so that logarithmic divergence of $g_1^{Bog}(r)$ turned into power-law decay of $g_1^{QC}(r)$. 
TEMPORAL COHERENCE
GENERAL CONSIDERATIONS

• If weak fluctuations of \( \hat{n}_0 \):
\[
\langle a_0^\dagger(t)a_0(0) \rangle \simeq \langle \hat{n}_0 \rangle \langle e^{-i[\hat{\theta}(t) - \hat{\theta}(0)]} \rangle
\]

• If phase change \( \hat{\theta}(t) - \hat{\theta}(0) \) has Gaussian distribution:
\[
\left| \langle a_0^\dagger(t)a_0(0) \rangle \right| \simeq \langle \hat{n}_0 \rangle e^{-\text{Var} [\hat{\theta}(t) - \hat{\theta}(0)]/2}
\]

• In terms of correlation function \( C(t) = \langle \dot{\theta}(t)\dot{\theta}(0) \rangle - \langle \dot{\theta} \rangle^2 \):
\[
\text{Var} [\hat{\theta}(t) - \hat{\theta}(0)] = 2t \int_0^t d\tau \ C(\tau) - 2 \int_0^t d\tau \ \tau C(\tau)
\]

<table>
<thead>
<tr>
<th>ballistic regime</th>
<th>diffusive regime</th>
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<tbody>
<tr>
<td>( \lim_{\tau \to +\infty} C(\tau) \neq 0 )</td>
<td>( C(\tau) = o(1/\tau) )</td>
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<tr>
<td>( \text{Var} [\hat{\theta}(t) - \hat{\theta}(0)] \sim At^2 )</td>
<td>( \text{Var} [\hat{\theta}(t) - \hat{\theta}(0)] \sim 2Dt )</td>
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TWO CASES DEPENDING ON $C(t \rightarrow +\infty)$

$\varphi = \hat{\theta}(t) - \hat{\theta}(0)$

**Diffusive:**

$\Delta \varphi^2 \sim 2t \int_0^{\infty} C(\tau) \, d\tau$

**Ballistic:**

$\Delta \varphi^2 \sim A \, t^2$
GENERAL CONSIDERATIONS (2)

Previous studies at $T > 0$:

- Sinatra, Witkowska, Castin (2006-): Clarification and quantitative studies.

Two key actors:

- Bogoliubov procedure eliminating the condensate mode from the Hamiltonian:

  $$H = E_0(N) + \sum_{k \neq 0} \epsilon_k \hat{b}_k^\dagger \hat{b}_k + H_3 + \ldots$$

  where $\epsilon_k$ is the Bogoliubov spectrum. Hamiltonian $H_3$ is cubic in field $\hat{\Lambda}$. It breaks integrability and plays central role in condensate dephasing (Beliaev-Landau pro-
cesses): \[
H_3 = g_0 \rho^{1/2} \sum_r b^3 \hat{\Lambda}^\dagger (\hat{\Lambda} + \hat{\Lambda}^\dagger) \hat{\Lambda}
\]

Quasi-particle resonant interactions à la Beliaev \( \hat{b}^\dagger \hat{b}^\dagger \hat{b} \)
and à la Landau \( \hat{b}^\dagger \hat{b} \hat{b} \): finite lifetime, kinetic equations
on mean quasi-particle occupation numbers

• Time derivative of condensate phase operator:

\[
\dot{\theta} \equiv \frac{1}{i\hbar} [\theta, H] \simeq -\mu_{T=0}(N)/\hbar - \frac{g_0}{\hbar L^3} \sum_{k \neq 0} (U_k + V_k)^2 \hat{n}_k
\]

with \( \hat{n}_k = \hat{b}_k^{\dagger} \hat{b}_k \). This contradicts Graham, 1998 and 2000. Keep in mind useful “magic” relation:

\[
\frac{g_0}{L^3} (U_k + V_k)^2 = \partial_N \epsilon_k
\]
Case of a pure condensate

- **One-mode model**, with $\hat{n}_0 = \hat{N}$: $H_{\text{one mode}} = \frac{g}{2L^3}\hat{N}^2$

- **Evolution of the condensate phase**:

\[
\dot{\theta}(t) = \frac{1}{i\hbar}[\hat{\theta}, H_{\text{one mode}}] = -\frac{g\hat{N}}{\hbar L^3} = -\mu(\hat{N})/\hbar
\]

- **No phase spreading** if fixed $N$.

- **Ballistic spreading** if $N$ fluctuates (Sols, 1994; Walls, 1996; Lewenstein, 1996; Castin, Dalibard, 1997)

\[
\text{Var} [\hat{\theta}(t) - \hat{\theta}(0)] = \left(\frac{t}{\hbar}\right)^2 \left(\frac{d\mu}{dN}\right)^2 \text{Var} \hat{N}
\]

- **Experiments**: Seen not for $\langle a_0^\dagger(t)a_0 \rangle$ but for $\langle a_0^\dagger(t)b_0(t) \rangle$ by interfering two condensats with common $t = 0$ phase [Bloch, Hänsch (2002); Pritchard, Ketterle (2006); Reichel, 2010.]
$T > 0$ gas prepared in the canonical ensemble

By analogy with previous case (Sinatra et al, 2007):

- As $N$, the energy $E$ is a constant of motion.
- Canonical ensemble = statistical mixture of eigenstates, $\text{Var } E \neq 0$ but $\text{Var } E \ll \bar{E}^2$ for a large system
- $\hat{\theta}(t) \sim -\mu mc (\hat{H})t/\hbar$ and weak fluctuations of $\hat{H}$:

$$\text{Var } [\hat{\theta}(t) - \hat{\theta}(0)] \sim (t/\hbar)^2 \left[ \frac{d\mu mc}{dE} (\bar{E}) \right]^2 \text{Var } E$$
From quantum ergodic theory (Sinatra et al, 2007):

- **Time average:**
  \[
  \langle \langle \dot{\theta}(t) \dot{\theta}(0) \rangle \rangle_t = \sum_{\lambda} \frac{e^{-\beta E_\lambda}}{Z} (\langle \Psi_\lambda | \dot{\theta} | \Psi_\lambda \rangle)^2
  \]

- **Deutsch (1991):** eigenstate thermalisation hypothesis. Mean value of observable $\hat{O}$ in one eigenstate $\Psi_\lambda$ very close to microcanonical value:
  \[
  \langle \Psi_\lambda | \hat{O} | \Psi_\lambda \rangle \simeq \bar{O}_{mc}(E = E_\lambda)
  \]

- $\hat{O} = \dot{\theta}$ in Bogoliubov limit: $\bar{\dot{\theta}}_{mc} = -\mu_{mc}/\hbar$.

- Linearize around mean energy due to weak (relative) energy fluctuations:
  \[
  \mu_{mc}(E_\lambda) \simeq \mu_{mc}(\bar{E}) + (E_\lambda - \bar{E}) \frac{d\mu_{mc}}{dE}(\bar{E})
  \]
Implications of previous result (canonical ensemble)

• The correlation function $C(\tau)$ of $\dot{\theta}$ does not tend to zero when $\tau \to +\infty$. Neither does the one of $\hat{n}_0$.

• This qualitatively contradicts Zoller, Gardiner, Graham. In qualitative agreement with Kuklov, Birman.

• Ergodicity ensured by interactions (cf. $H_3$) among Bogoliubov quasi-particles.

• Approximating $H$ with integrable $H_{\text{Bog}}$, as eventually done by Kuklov and Birman, gives incorrect coefficient of $t^2$.

Why failure of master equation method of Zoller-Gardiner?

Setting $\hat{n}_k \equiv \hat{b}_k^\dagger \hat{b}_k$: $C(t) = \sum_{k,k'} A_k A_{k'} \langle \delta \hat{n}_k(t) \delta \hat{n}_{k'}(0) \rangle$

Master equation + quantum regression theorem:

- System = Bogoliubov modes $k$ and $k'$. Other modes = reservoir. Born-Markov approximation:
  $$\langle \delta \hat{n}_k(t) \delta \hat{n}_{k'}(0) \rangle = \delta_{kk'} \bar{n}_k (1 + \bar{n}_k) e^{-\Gamma_k t}$$
  so $C(t) \xrightarrow{t \to \infty} 0$ and phase has diffusive spreading...

But reservoir not truly infinite:

- From ergodic theory:
  $$\langle \delta \hat{n}_k(t) \delta \hat{n}_{k'}(0) \rangle \xrightarrow{t \to \infty} \frac{\epsilon_k \bar{n}_k (\bar{n}_k + 1) \epsilon_{k'} \bar{n}_{k'} (\bar{n}_{k'} + 1)}{\sum_{q \neq 0} \epsilon_q^2 \bar{n}_q (1 + \bar{n}_q)} \propto \frac{1}{V}$$
  and double sum: $VC(t) \xrightarrow{t \to \infty} 0$ even in thermodynamic $V \to \infty$ limit.
Illustration with a classical field calculation

Figure 1: For a gas prepared in canonical ensemble, correlation function of $\dot{\theta}$ for the classical field. The equation of motion is the non-linear Schrödinger equation. A. Sinatra, Y. Castin, E. Witkowska, Phys. Rev. A 75, 033616 (2007).
Gas prepared in the microcanonical ensemble: phase diffusion

- The conserved quantities $N, E$ do not fluctuate. One finds $C(\tau) = O(1/\tau^3)$ and $\text{Var} [\hat{\theta}(t) - \hat{\theta}(0)] \sim 2Dt$.

- One needs the full dependence of $C(\tau)$ to get $D$.

- As we have seen, $C(\tau)$ can be deduced from all the $\langle \hat{n}_k(\tau)\hat{n}_{k'}(0) \rangle$.

- The gas is in a statistical mixture of Fock states quasi-particles $|\{n_q\}\rangle$. One simply needs $\langle \{n_q\}|\hat{n}_k(\tau)|\{n_q\}\rangle$.

- The evolution of the mean number of quasi-particles is given by quantum kinetic equations including the Beliaev-Landau processes due to $H_3$. 
The quantum kinetic equations

\[ \dot{n}_q = -\frac{g^2 \rho}{\hbar \pi^2} \int d^3k \left\{ n_q n_k - n_{q+k}(1 + n_k + n_q) \right\} \left( A_{k,q}^{q+k} \right)^2 \times \delta(\epsilon_q + \epsilon_k - \epsilon_{q+k}) \right\} 

- \frac{g^2 \rho}{2\hbar \pi^2} \int d^3k \left\{ n_q(1 + n_k + n_{q-k}) - n_k n_{q-k} \right\} \left( A_{k,q}^q \right)^2 \times \delta(\epsilon_k + \epsilon_{q-k} - \epsilon_q) \right\} 

with the Beliaev-Landau coupling amplitudes:

\[ A_{k,k'}^q = U_q U_k U_{k'} + V_q V_k V_{k'} + (U_q + V_q)(V_k U_{k'} + U_k V_{k'}) \cdot \]

Diffusion coefficient of the condensate phase

Figure 2: Universal result in Bogoliubov limit (weakly interacting, $T \ll T_c$).

Summary of results for the phase spreading

\[ \text{Var} \left[ \theta(t) - \theta(0) \right] \underset{t \to +\infty}{=} \text{Var} (E) \left[ \frac{d\mu_{mc}}{\hbar dE} (\bar{E}) \right]^2 t^2 + 2Dt + c + O\left(\frac{1}{t}\right) \]

- Coefficient of \( t^2 \) depends on the ensemble. First obtained with quantum ergodic theory (Sinatra, Castin, Witkowska, 2007) but also with quantum kinetic theory (from existence of undamped mode of linearized kinetic equations due to energy conservation). Interpretation:

\[ \theta(t) - \theta(0) \underset{t \to +\infty}{\sim} -\mu(H)t/\hbar. \]

- Diffusion coefficient \( D \) is ensemble independent. \( \hbar DL^3/g \) function of \( k_BT/\rho g \) (Sinatra, Castin, Witkowska, 2009).
- Ensemble independent \( c \neq 0 \): \( C_{mc}(t) \) not a Dirac.
AN EXAMPLE FOR $k_B T = 10 \rho g$
Our publications on the subject

LIMIT OF SPIN SQUEEZING IN FINITE TEMPERATURE
BOSE-EINSTEIN CONDENSATES

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ATOMIC CLOCKS IN BRIEF

What an atomic clock does:

- Measures the transition frequency $\omega_{ab}$ of two-level atoms
- Formally, a two-level atom is a spin $1/2$
- Collective spin $S = \sum_{i=1}^{N} S_i$, free Hamiltonian:
  $$H_0 = \hbar \omega_{ab} S_z$$
- At time $0$, prepare the collective spin along $x$. At time $\tau$, measurement of the spin precession angle $\omega_{ab}\tau$ gives transition frequency $\omega_{ab}$ (Ramsey method).

Transverse quantum fluctuations: $\Delta S_y \Delta S_z \geq \frac{1}{2} |\langle S_x \rangle|$

- Standard quantum limit: All spins along $x$, $\langle S_x \rangle = N/2$:
  $$\Delta S_y^{st} = \Delta S_z^{st} = \sqrt{N}/2 \quad \rightarrow \quad \Delta \omega_{ab} = \frac{1}{N^{1/2} \tau}$$
- This is larger than technical noise in good clocks
ONE CAN GAIN WITH SPIN SQUEEZED STATES

- Can reduce a lot $\Delta S_y$, at the expense of increasing $\Delta S_z$
- Gain $1/\xi$ on signal-to-noise ratio (Wineland, 1994):

$$\xi^2 = \frac{N \Delta S_y^2}{\langle S_x \rangle^2} < 1 \longrightarrow \Delta \omega_{ab} = \frac{\xi}{N^{1/2}}$$

Kitagawa-Ueda spin squeezing: $H = \hbar \omega_{ab} S_z + \hbar \chi S_z^2$

- Spin-dependent Larmor frequency: Evolution turns the fluctuation circle into a tilted ellipse. At best time:

$$\xi_{\min}^2 \sim \frac{3^{2/3}}{2N^{2/3}} \quad \text{as } N \to \infty$$

- Realisable with two-mode condensates (Cirac, 2001):

$$S_x + iS_y = a^\dagger b, \quad S_z = (a^\dagger a - b^\dagger b)/2, \quad \chi = \frac{g}{\hbar V}$$

- In the lab (Oberthaler, 2008; Treutlein, 2010): $1/\xi = 3$
ON THE BLOCH SPHERE

In practice, squeezed axis is tilted (rotation required):

$$\Delta S_{\perp, \text{min}}^2 = \frac{1}{2} \left[ \langle S_y^2 \rangle + \langle S_z^2 \rangle - |\langle (S_y + iS_z)^2 \rangle| \right]$$
WHAT HAPPENS IN REAL LIFE?

An atomic gas is a multimode system:

- At $T > 0$, the non-condensed modes constitute a dephasing environment for the condensate: phase spreading and a finite coherence time

- What is the effect on spin squeezing?

From Bogoliubov theory in brief:

- Much before the phase collapse time, $\rho g t / \hbar \ll N^{1/2}$:
  
  $$\langle S_x \rangle \simeq \frac{N}{2}, \quad S_y \simeq -\frac{N}{2} (\theta_a - \theta_b), \quad S_z = \frac{N_a - N_b}{2}$$

- Evolution of phase operators (previous lecture):
  
  $$\left( \hat{\theta}_a - \hat{\theta}_b \right)(t) = \left( \hat{\theta}_a - \hat{\theta}_b \right)(0^+) - \frac{gt}{\hbar V} \left[ 2S_z + D \right]$$

  $$D = \sum_{k \neq 0} (U_k + V_k)^2 (\hat{n}_{ak} - \hat{n}_{bk}) \left[ \hat{n}_{\sigma k} = \text{quasi-particle number} \right]$$
In thermodynamic limit:

\[
\xi^2(t) = \frac{1}{(\tau + \sqrt{1 + \tau^2})^2} + \frac{2\langle D^2 \rangle N \tau^2}{(\tau + \sqrt{1 + \tau^2})\sqrt{1 + \tau^2}}
\]

with reduced time \( \tau = \rho g t / (2 \hbar) \).

- First term is Kitagawa-Ueda model.
- Second term saturates to minimal squeezing:

\[
\xi_{\text{min}}^2 = \frac{\langle D^2 \rangle}{N} = (\rho a^3)^{1/2} f(k_B T / \rho g)
\]
$\xi^2(t)$ FOR BOGOLIUBOV THEORY

$(\rho a^3)^{1/2} = 10^{-3}, k_B T / \rho g = 1$
MINIMAL $\xi^2$ FOR BOGOLIUBOV THEORY

\[ k_B T/\rho g \]

\[ \xi_{\text{min}}^2 \sim \frac{\langle N_{nc} \rangle / N}{(\rho a^3)^{1/2}} \]
VALIDITY CONDITIONS

- System out-of-equilibrium after pulse
- Will thermalize, this is neglected in Bogoliubov theory
- Have the close-to-best-squeezing time

\[ \frac{\rho g t_\eta}{\hbar} \simeq \frac{1}{\eta^{1/2} \xi_{\text{min}}} \]

smaller than thermalisation time, estimated by Beliaev-Landau damping rates of modes of energy \( k_B T \) or \( \rho g \):

\[ \frac{\rho g t_{\text{therm}}}{\hbar} \propto \frac{1}{(\rho a^3)^{1/2}} \]

- Validity condition satisfied in weakly interacting limit:

\[ \frac{t_\eta}{t_{\text{therm}}} \propto (\rho a^3)^{1/4} \ll 1 \]
Summary of results for spin squeezing:

• For atoms with two internal states $a$ and $b$, apply a $\pi/2$ pulse on a condensate initially in $a$. Due to interactions, phase state transformed into spin squeezing state.

• If injected in an atomic clock, statistical uncertainty on clock frequency after interrogation time $\tau$:

$$\Delta \omega_{ab} = \frac{\Delta S_{\perp, \text{min}}}{\langle S_x \rangle \tau} \equiv \frac{\xi}{N^{1/2} \tau}$$

• Spin dynamics is a phase dynamics: $S_z = \text{const}$, $S_x \approx \text{const}$,

$$S_y \propto \theta_a - \theta_b \propto (N_a - N_b + D)t \delta N \mu/\hbar$$

where $D$ due to multimode nature of the fields (random dephasing environment). Best squeezing in weakly interacting, thermodynamic limit does not vanish:

$$\xi_{\text{min}}^2 \approx \frac{\langle D^2 \rangle}{N}$$
Our publications on the subject


THREE-DIMENSIONAL IMPURITY PROBLEMS
WITH COLD ATOMS

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A RICH PROBLEM

An impurity atom $A$ (mass $m_A$) interacting with another species (or spin state) $B$ (mass $m_B$) [no interaction among $B$ atoms]:

1. monomeron/dimeron pb: a Fermi sea of $B$ atoms on a (narrow) Feshbach resonance [with C. Trefzger]

2. strong localisation of $A$: the $B$ atoms are randomly pinned at the nodes of an optical lattice [with M. Antezza, D. Hutchinson, P. Massignan, U. Gavish]

3. photonic band gaps: $A$ is a photon; one $B$ atom per node of an optical lattice [with M. Antezza]
1 Monomeron-dimeron problem

1.1 Physical motivation

• Monomerons and dimerons are quasi-particles belonging to the general class of Fermi polarons [neutral objects dressed by the Fermi sea, rather than electrons dressed by phonons in solids]

• The ground state of strongly polarized spin 1/2 Fermi gas at unitarity \([N_A \ll N_B, A = \downarrow, B = \uparrow]\) is a Fermi gas of monomerons [Chevy; Lobo, Recati, Giorgini, Stringari; Combescot, Giraud, Leyronas; Mora]

• monomeron = \(A\) dressed by particle-hole excitations of \(B\) Fermi sea. It is a quasi-particle of dispersion relation

\[
\Delta E(P) \underset{P \to 0}{=} \Delta E(0) + \frac{P^2}{2m_*} + O(P^4)
\]
At unitarity, measured equation of state of monomerons \( \simeq \) an ideal Fermi gas [Navon, Nascimbène, Chevy, Salomon]

1.2 Monomeron vs dimeron

- The ground state of \( A \), as the function of the \( AB \) scattering length \( a \), has two branches with a cusp [Prokof’ev, Svistunov]

- Dimeron = \( AB \) dimer dressed by particle-hole excitations of \( B \) Fermi sea

- transition from monomeron to dimeron at \( a_c \)

- \( a_c > 0 \) (broad Feshbach resonance) “intuitive”: dimer exists in free space for \( a > 0 \) only.
1.3 Two-channel model

\[ H = \sum_k \left( \frac{\hbar^2 k^2}{2m_A} a_k \dagger a_k + \frac{\hbar^2 k^2}{2m_B} b_k \dagger b_k + (E_{\text{mol}} + \frac{\hbar^2 k^2}{2(m_A + m_B)}) \gamma_k \dagger \gamma_k \right) + \frac{\Lambda}{L^{3/2}} \sum_{k_A, k_B} \chi(k_{\text{rel}})[\gamma_{k_A} \dagger + k_B a_{k_A} b_{k_B} + \text{h.c.}] \]

Take cut-off to \( \infty \) for \( a \) and \( \Lambda \) fixed \((E_{\text{mol}} \rightarrow +\infty)\). Feshbach length \( R_\ast = \pi \hbar^4 / (\mu^2 \Lambda^2) \). Free-space dimer iff \( a > 0 \).

1.4 How to solve

For \( \Lambda = 0 \):

one \( A \), zero molecule \quad zero \( A \), one molecule

\[ E_0 = E_{\text{FS}}(N_B) \quad E_0 = E_{\text{FS}}(N_B - 1) + E_{\text{mol}} \]

\[ \Delta E_{\text{pol}} = 0 \quad \Delta E_{\text{dim}} = E_{\text{mol}} - E_F \]

For \( \Lambda > 0 \): Expand in the number of particle-hole pairs [Chevy; Combescot, Giraud], here up to one pair.
Generalized Chevy Ansätze

|ψ_{mono}\rangle =

B: \begin{cases} 
\text{Mol:vacuum} & k_A = 0 \\
H & k_{mol} = q \\
\text{vacuum} & \text{vacuum}
\end{cases}

A: k_A = q-k

|ψ_{dim}\rangle =

B: \begin{cases} 
\text{Mol:k}_{mol} = 0 & k_A = -k \\
H & k_{mol} = q-k \\
\text{vacuum} & \text{vacuum}
\end{cases}

A: vacuum

k_A = q-(k+k')

\text{vacuum

\text{vacuum}
$a_c < 0$ at large $k_F R^*_\ast$. Paradoxical. Stable dimeronic branch extends to a regime where no free space dimer.
1.6 Analytics

- Two-body scattering amplitude [Petrov]:
  \[ f_{k_{\text{rel}}} = -\frac{1}{\frac{1}{a} + ik_{\text{rel}} + k_{\text{rel}}^2 R_*} \]

- Usual weakly attractive limit: \( a \to 0^-, R_* \) fixed

- Appropriate weakly attractive limit: \( a \to 0^-, aR_* \) fixed

- Define \( \alpha = \frac{m_A}{m_A + m_B} \) and

  \[ s \equiv \alpha k_F (-a R_*)^{1/2} \]

Note that \( s \to 0 \) in the weakly attractive limit \( a \to 0^- \).
\[
\lim_{a \to 0} \Delta E/E_F = \frac{m_B}{m_A} \frac{N_{\text{mol}}=0}{N_{\text{mol}}=1}
\]

\[
\frac{m_A}{m_A + m_B} \left[ \frac{m_A}{m_A + m_B} \right]^{1/2}
\]

\[
S
\]
Critical scattering length on a narrow Feshbach resonance:

\[
\frac{1}{k_F a_c} \overset{k_F R_* \to \infty}{=} -\alpha k_F R_*
\]

\[
+ \frac{2}{\pi} \left[ 1 - \alpha^{-2} + \frac{1}{2} \left( \alpha^{-5/2} - \alpha^{1/2} \right) \ln \frac{1 + \alpha^{1/2}}{1 - \alpha^{1/2}} \right] + O\left( \frac{1}{k_F R_*} \right)
\]
1.7 Physical interpretation of $1/(k_F a)_c < 0$

- **Stabilization of molecule by Fermi sea**
- **Molecule energy renormalized by Lambshift:**
  \[
  \tilde{E}_{\text{mol}} = E_{\text{mol}} + \int \frac{d^3k}{(2\pi)^3} \frac{\chi^2(k)\Lambda^2}{0 - \hbar^2 k^2 / 2\mu} = -\frac{\Lambda^2 \mu}{2\pi \hbar^2 a}
  \]
- **Free space:** molecule stable iff $\tilde{E}_{\text{mol}} < 0$ that is $a > 0$
- **Fermi sea:** molecule stable iff $\tilde{E}_{\text{mol}} < E_F$ that is
  \[
  \frac{1}{k_F a} > -\alpha k_F R_*
  \]
- **For** $a < 0$ **this imposes**
  \[
  s > \alpha^{1/2} = \left(\frac{m_A}{m_A + m_B}\right)^{1/2}
  \]
2 Strong (Anderson) localisation of $A$ matterwave

2.1 Physical motivation and configuration

- $B$ randomly filling the nodes of an optical lattice ($p_{occ} \ll 1$) with no tunneling
- $A$ does not see the lattice potential, it sees a disordered ensemble of scatterers
- $\hbar^2 k_A^2/2m_A \ll \hbar \omega_B$ so elastic $AB$ scattering
• A solvable alternative to laser speckle (Aspect). Optimisation of localisation by tuning the $A - B$ scattering length

• One expects (in 3D) an Anderson transition [mobility edge] between extended states (continuous spectrum) and localized states (point-like spectrum).

2.2 Model

• Each trapped $B$ atom replaced by Wigner-Bethe-Peierls contact conditions at lattice node on $A$ wavefunction, with effective scattering length $a_{\text{eff}}$:

$$\phi(r_A) \overset{r_{AB} \to 0}{=} D(r_B) \times \left( \frac{1}{r_{AB}} - \frac{1}{a_{\text{eff}}} \right) + O(r_{AB})$$

zero energy scattering state for $r_{AB} \gg a_{\text{ho}}$
Scatterers occupy a sphere of finite but large diameter

2.3 How to solve

• single particle Green’s function in presence of a number $N_B$ of $B$ scatterers at energy $E = \hbar^2 k_A^2 / 2m_A$ exactly given by $N_B \times N_B$ matrix inversion

\[
G(r, r_0) = g(r - r_0) + \frac{2\pi \hbar^2}{m_A} \sum_{i,j=1}^{N_B} g(r - r_i)[M^{-1}]_{ij}g(r_j - r_0)
\]

\[
M_{ij} = \begin{cases} 
-\frac{2\pi \hbar^2}{m} g(r_i - r_j) = e^{ik_A r_{ij}}/r_{ij} & \text{if } i \neq j, \\
ik_A + a_{\text{eff}}^{-1} & \text{if } i = j.
\end{cases}
\]

where free-space Green’s function $g(r - r_0)$ is translationally invariant.
• Gives access to localisation length $\xi$: decay length of field radiated by a source of $A$ in the $B$ medium:

$$G(r, r_0) \sim A \frac{e^{-|r-r_0|/\xi}}{|r - r_0|^{\alpha}}$$

large $|r-r_0|$

• Gives access to density $\rho_E$ of states ($E < 0$) [real poles of $G$] or resonances [complex poles of $G$] ($E = \text{Re } z_0 > 0$, $\hbar \Gamma / 2 = - \text{Im } z_0 > 0$)

2.4 Matrix $M$ is not an Anderson problem

• off-diagonal disorder only

• at $E > 0$, $M_{ij}$ decays slowly for $r_{ij} \to \infty$: more slowly than $1/r_{ij}^3$. So no localized states? (Boris Altshuler, informal discussion)
2.5 Numerical results

- $p_{occ} = 1/10$, diameter $= 140d$; $\langle N_B \rangle \simeq 1.4 \times 10^5$
- for $a_{\text{eff}} < 0$, for $a_{\text{eff}} \to +\infty$, no evidence of localisation [van Tiggelen, Lagendijk]
- The best for localisation is to take $a_{\text{eff}} \simeq$ half mean inter-$B$ distance
- Rich situation because we consider also $E < 0$. E.g. for $a_{\text{eff}} = d$ we shall find three mobility edges, one at positive energy and two at negative energy. Is certainly sensitive to the presence of the lattice.
\( a_{\text{eff}}/d = 0.1 \) (black), 0.2 (red), 0.7 (green), 1 (blue), 1.3 (violet)
Oblique line = mean-field bottom \( E = \rho B g_{\text{eff}} \)
The diagram illustrates the behavior of a system as a function of $E/E_0$. The x-axis represents $E/E_0$ ranging from $-1.3$ to $0.6$, and the y-axis represents $\xi/d$ ranging from $0$ to $12$. The figure is divided into different regions:

1. **Spectral Gap** ($\rho_E = 0$) at $E/E_0 = -1.3$:
   - Localized states ($\rho_E^\text{loc} > 0$) with $\rho_E > 0$.
   - Extended states ($\rho_E^\text{loc} = 0$) with $\rho_E > 0$.

2. **Extended States** ($\rho_E > 0$) at $E/E_0 = 0.6$:
   - Localized states ($\rho_E^\text{loc} > 0$).
   - Extended states ($\rho_E^\text{loc} = 0$).

The effective mass $a_{\text{eff}} = d$ is indicated at the top right of the diagram.
3 Photonic band gaps

3.1 Physical motivation

- find matter lattices leading to omnidirectional band gaps (OBG) for light
- first studies with extended objects (dielectric spheres) [Ho, Cha Soukoulis, 1990]
- for atoms: Lagendijk et al., 1996 predict OBG for fcc lattice
- Knoester et al., 2006: Lagendijk’s sum is divergent; addition by hand of a regularizing term; no OBG for fcc. But this no longer solves the original Hamiltonian problem.
3.2 Case of classical physics

- for a stationary state of the field at frequency $\omega$, atom in $R$ carries a mean electric dipole $\vec{D}(R)$:

$$\vec{D}(R) = \varepsilon_0 \alpha(\omega) \sum_{R' \neq R} g(R - R') \vec{D}(R')$$

- $\alpha(\omega) \propto 1/(\omega - \omega_0 + i\Gamma/2)$ is atomic polarisability

- $g_{ij}(r)$ is component along $i$ of electric field radiated in $r$ by a unit dipole oscillating along $j$ and located at the origin of coordinates:

$$g_{ij}(r) \propto [(\omega/c)^2 \delta_{ij} + \partial_r \partial_r] \frac{e^{i(\omega/c)r}}{r}$$

ensures transversality scalar field
• periodic case, Bloch theorem: \( \hbar \omega = \epsilon_q \) with
\[
\vec{D}(R) = \vec{D} e^{iq \cdot R}
\]

3.3 How to calculate the sum?

• “formule sommatoire de Poisson” :
\[
\sum_{R \in DL} f(R) = \frac{1}{V} \sum_{K \in RL} \tilde{f}(K)
\]

• but \( g(0) = \infty \) and \( \sum_K K_i K_j / K^2 \) not absolutely convergent

• Physical regularising effect: atomic positions delocalized over \( a_{ho} \), provides a Gaussian cut-off in Fourier space
\[
g(R - R') \rightarrow \langle g(R + u - R' - u') \rangle_{u,u'} = \bar{g}(R - R')
\]
3.4 Looking for band gaps

- none for Bravais lattices (Knoester was right)
- the historical work of Soukoulis predicted OBG for a diamond lattice of dielectric spheres (fcc+translated copy by \((d/4, d/4, d/4)\)). May be it also works with atoms?
- we indeed predict a gap for an atomic diamond lattice
- important to check for the absence of free wave (field vanishes on all lattice sites) using

\[
\frac{\omega_{\text{free}}}{c} \geq \inf_{K \neq 0} \frac{K}{2}
\]
Photonic density of states for diamond, $k_0d = 2 \ (k_0 = \omega_0/c)$
diamant aho=0.08 ; extrapolated to null range
THE UNITARY GAS:
SYMMETRY PROPERTIES AND APPLICATIONS

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LKB and LPA, Ecole normale supérieure (Paris, France)

Ludovic Pricoupko
LPTMC, Université Paris 6
GENERAL CONTEXT

The physical system:

- Fermionic atoms with two internal states $\uparrow, \downarrow$
- Short-range interactions between $\uparrow$ and $\downarrow$ controlled by a magnetic Feshbach resonance
- Arbitrary values for the numbers $N_\uparrow$, $N_\downarrow$
- Intense experimental studies (Thomas, Salomon, Jin, Ketterle, Grimm, Hulet, Zwierlein...), e.g. BEC-BCS crossover (Leggett, Nozières, Schmitt-Rink, Sa de Melo,...)

What is not discussed here:

- The actual many-body state of the system: superfluid or normal
- The particularly intriguing strongly polarized case $N_\uparrow \gg N_\downarrow$: Polaronic physics
OUTLINE OF THE TALK

• What is the unitary gas?
• Simple consequences of scaling invariance
• Dynamical consequences: $SO(2,1)$ hidden symmetry in a trap
• Separability in hyperspherical coordinates
• Does the unitary gas exist?
WHAT IS THE UNITARY GAS?
DEFINITION OF THE UNITARY GAS

• Opposite spin two-body scattering amplitude

\[ f_k = -\frac{1}{ik} \quad \forall k \]

• “Maximally” interacting: Unitarity of \( S \) matrix imposes \( |f_k| \leq 1/k \).

• In real experiments with magnetic Feshbach resonance:

\[ \frac{1}{f_k} = \frac{1}{a} + ik - \frac{1}{2}k^2r_e + O(k^4b^3) \]

unitary if “infinite” scattering length \( a \) and “zero” ranges:

\[ k_{\text{typ}}|a| > 100, \quad k_{\text{typ}}|r_e| \quad \text{and} \quad k_{\text{typ}}b < \frac{1}{100} \]

imposing \( |a| > 10 \) microns for \( r_e \sim b \sim \) a few nm.

• All these two-body conditions are only necessary.
THE ZERO-RANGE WIGNER-BETHE-Peierls MODEL

- Interactions are replaced by contact conditions.
- For $r_{ij} \to 0$ with fixed $ij$-centroid $\vec{C}_{ij} = (\vec{r}_i + \vec{r}_j)/2$ different from $\vec{r}_k, k \neq i, j$:
  \[
  \psi(\vec{r}_1, \ldots, \vec{r}_N) = \left( \frac{1}{r_{ij}} - \frac{1}{a} \right) A_{ij}[\vec{C}_{ij}; (\vec{r}_k)_{k \neq i,j}] + O(r_{ij})
  \]
- Elsewhere, non interacting Schrödinger equation
  \[
  E\psi(\vec{X}) = \left[ -\frac{\hbar^2}{2m} \Delta + \frac{1}{2}m\omega^2 X^2 \right] \psi(\vec{X})
  \]
  with $\vec{X} = (\vec{r}_1, \ldots, \vec{r}_N)$.
- Odd exchange symmetry of $\psi$ for same-spin fermion positions.
- Unitary gas exists iff Hamiltonian is self-adjoint.
EXERCISING WITH THE BETHE-PEIERLS MODEL

Scattering state of two particles:

\[ \phi_k(r) = e^{ik \cdot r} + f_k \frac{e^{ikr}}{r} \]

• For \( r > 0 \) this is an eigenstate of the non-interacting problem.

• Contact condition in \( r = 0 \):

\[ \frac{f_k}{r} + (1 + ikf_k) + O(r) = \frac{A}{r} + O(r) \]

determines scattering amplitude \( f_k \):

\[ f_k = -\frac{1}{ik} \]
SIMPLE CONSEQUENCES OF SCALING INVARIANCE
SCALING INVARIANCE OF CONTACT CONDITIONS

\[
\psi(\vec{X}) = \frac{1}{r_{ij} \rightarrow 0} A_{ij} [\tilde{C}_{ij}; (\vec{r}_k)_{k \neq i,j}] + O(r_{ij})
\]

- Domain of Hamiltonian is scaling invariant: If \( \psi \) obeys the contact conditions, so does \( \psi_\lambda \) with

\[
\psi_\lambda(\vec{X}) \equiv \frac{1}{\lambda^{3N/2}} \psi(\vec{X}/\lambda)
\]

- Consequences (also true for the ideal gas):

<table>
<thead>
<tr>
<th>free space</th>
<th>box (periodic b.c.)</th>
<th>harm. trap</th>
</tr>
</thead>
<tbody>
<tr>
<td>no bound state</td>
<td>( PV = 2E/3 )</td>
<td>virial ( E = 2E_{\text{harm}} )</td>
</tr>
</tbody>
</table>

(*) If \( \psi \) of eigenenergy \( E \), \( \psi_\lambda \) of eigenenergy \( E/\lambda^2 \). Square integrable eigenfunctions (after center of mass removal) correspond to point-like spectrum, for selfadjoint \( H \). (***) \( E(N, V \lambda^3, S) = E(N, V, S)/\lambda^2 \), then take derivative in \( \lambda = 1 \). (***) For eigenstate \( \psi \), mean energy of \( \psi_\lambda \),

\[
E_\lambda = \frac{\langle H_{\text{Laplacian}} \rangle}{\lambda^2} + \langle H_{\text{harm}} \rangle \lambda^2
\]
stationary in \( \lambda = 1 \).
DYNAMICAL CONSEQUENCES:

$SO(2,1)$ HIDDEN SYMMETRY IN A TRAP
IN A TIME-DEPENDENT TRAP

- At \( t = 0 \): static trap \( U(r) = m\omega^2r^2/2 \), system in eigenstate \( \psi_0(\vec{X}) \) of energy \( E \).

- For \( t > 0 \), arbitrary time dependence of trap spring constant, \( \omega(t) \). Known solution for ideal gas:

\[
\psi(\vec{X}, t) = \frac{e^{-i\theta(t)}}{\lambda^3N/2(t)} \exp \left[ \frac{im\ddot{\lambda}}{2\hbar\lambda} X^2 \right] \psi_0(\vec{X}/\lambda(t))
\]

with \( \ddot{\lambda} = \omega^2\lambda^{-3} - \omega^2(t)\lambda \) and \( \dot{\theta} = E\lambda^{-2}/\hbar \).

- This is a gauge plus scaling transform.

- The gauge transform also preserves contact conditions:

\[
\sum_{i,j} r_i^2 + r_j^2 = 2C_{ij}r_{ij}^2 + \frac{1}{2}r_{ij}^2
\]

so solution also applies to unitary gas!

IN THE MACROSCOPIC LIMIT

\[ \psi(\vec{X}, t) = e^{-i\theta(t)} \frac{e^{-i\theta(t)}}{\lambda^{3N/2}} \exp \left[ \frac{im\dot{\lambda}}{2\hbar\lambda} X^2 \right] \psi_0(\vec{X}/\lambda) \]

<table>
<thead>
<tr>
<th>density ( \rho(\vec{r}, t) = \rho_0(\vec{r}/\lambda)/\lambda^3 )</th>
<th>velocity field ( \vec{v}(\vec{r}, t) = \vec{r}\dot{\lambda}/\lambda )</th>
</tr>
</thead>
<tbody>
<tr>
<td>local temp. ( T(\vec{r}, t) = T/\lambda^2 )</td>
<td>pressure ( P(\vec{r}, t) = P_0(\vec{r}/\lambda)/\lambda^5 )</td>
</tr>
<tr>
<td>local entropy per particle</td>
<td>( s(\vec{r}, t) = s_0(\vec{r}/\lambda) )</td>
</tr>
</tbody>
</table>

This has to solve the hydrodynamic equations for a normal gas. Entropy production equation:

\[
\rho k_B T (\partial_t s + \vec{v} \cdot \vec{\nabla} s) = \nabla \cdot (\kappa \nabla T) + \zeta (\vec{\nabla} \cdot \vec{v})^2 + \frac{\eta}{2} \sum_{i,j} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \vec{\nabla} \cdot \vec{v} \right)^2
\]

so the bulk viscosity is zero: \( \zeta(\rho, T) = 0 \ \forall T > T_c \). Reproduces the conformal invariance result of Son (2007).
LADDER STRUCTURE OF THE SPECTRUM

• Infinitesimal change of \( \omega \) for \( 0 < t < t_f \). For \( t > t_f \):
  \[
  \lambda(t) - 1 = \epsilon e^{-2i\omega t} + \epsilon^* e^{2i\omega t} + O(\epsilon^2)
  \]
  so an undamped mode of frequency \( 2\omega \).

• Corresponding wavefunction change:
  \[
  \psi(\vec{X}, t) = \left[ e^{-iEt/\hbar} - \epsilon e^{-i(E+2\hbar\omega)t/\hbar} L_+ + \epsilon^* e^{-i(E-2\hbar\omega)t/\hbar} L_- \right] \psi_0(\vec{X}) + O(\epsilon^2)
  \]

• Raising and lowering operators:
  \[
  L_\pm = \pm i \left[ \frac{3N}{2i} - i \vec{X} \cdot \partial \vec{X} \right] + \frac{H}{\hbar\omega} - m\omega X^2 / \hbar
  \]
  (in red, generator of scaling transform)

• Spectrum=collection of semi-infinite ladders of step \( 2\hbar\omega \).
  \( SO(2, 1) \) hidden symmetry (Pitaevskii, Rosch, 1997).
LADDER STRUCTURE OF THE SPECTRUM (2)

\[ E_g + 2\hbar\omega \]

\[ E_g + 4\hbar\omega \]

\[ E_g + 6\hbar\omega \]

\[ E_g + 8\hbar\omega \]
USEFUL MAPPING AND SEPARABILITY

- Each energy ladder has a ground step of energy $E_g$, eigenfunction $\psi_g$.
- Integration of $L_- \psi_g = 0$ gives, with $\vec{X} = X\vec{n}$:
  \[
  \psi_g(\vec{X}) = e^{-m\omega X^2/2\hbar} \times \left[ X\frac{E_g}{(\hbar\omega)} - \frac{3N}{2} f(\vec{n}) \right]
  \]
- Limit $\omega \rightarrow 0$: mapping to zero energy free space solutions. N.B.: $E_g/(\hbar\omega)$ is a constant.
- Free space problem solved for $N = 3$ (Efimov, 1972)... so trapped case also solved (Werner, Castin, 2006).
- Also, this is separable in hyperspherical coordinates [Werner, Castin, PRA 74, 053604 (2006)].
SEPARABILITY IN HYPERSPHHERICAL COORDINATES
SEPARABILITY IN INTERNAL COORDINATES

• Use Jacobi coordinates to separate center of mass $\vec{C}$

• Hyperspherical coordinates (arbitrary masses $m_i$):
  $$(\vec{r}_1, \ldots, \vec{r}_N) \leftrightarrow (\vec{C}, R, \vec{\Omega})$$
  with $3N - 4$ hyperangles $\vec{\Omega}$ and the hyperradius
  $$\bar{m}R^2 = \sum_{i=1}^{N} m_i (\vec{r}_i - \vec{C})^2$$
  where $\bar{m}$ is the mean mass.

• Hamiltonian is clearly separable:
  $$H_{\text{internal}} = -\frac{\hbar^2}{2\bar{m}} \left[ \frac{\partial^2}{\partial R^2} + \frac{3N - 4}{R} \partial_R + \frac{1}{R^2} \Delta \vec{\Omega} \right] + \frac{1}{2} \bar{m} \omega^2 R^2$$
Do the contact conditions preserve separability?

- For free space $E = 0$, yes, due to scaling invariance:
  \[ \psi_{E=0} = R^{s-(3N-5)/2} \phi(\tilde{\Omega}) \]
  $E = 0$ Schrödinger’s equation implies
  \[ \Delta_{\tilde{\Omega}} \phi(\tilde{\Omega}) = - \left[ s^2 - \left( \frac{3N - 5}{2} \right)^2 \right] \phi(\tilde{\Omega}) \]
  with contact conditions. $s^2 \in$ discrete real set.

- For arbitrary $E$, Ansatz with $E = 0$ hyperrangular part obeys contact conditions $[R^2 = R^2(r_{ij} = 0) + O(r_{ij}^2)]$:
  \[ \psi = F(R) R^{- (3N-5)/2} \phi(\tilde{\Omega}) \]

- Schrödinger’s equation for a fictitious particle in 2D:
  \[ EF(R) = - \frac{\hbar^2}{2\bar{m}} \Delta_{R}^{2D} F(R) + \left[ \frac{\hbar^2 s^2}{2\bar{m}R^2} + \frac{1}{2} \bar{m}\omega^2 R^2 \right] F(R) \]
SOLUTION OF HYPERRADIAL EQUATION \((N \geq 3)\)

\[
EF(R) = -\frac{\hbar^2}{2\bar{m}} \Delta_{R}^{2D} F(R) + \left[ \frac{\hbar^2 s^2}{2\bar{m}R^2} + \frac{1}{2}\bar{m}\omega^2 R^2 \right] F(R)
\]

- Which boundary condition for \(F(R)\) in \(R = 0\)? Wigner-Bethe-Peierls does not say.
- **Key point:** particular solutions \(F(R) \sim R^{\pm s}\) for \(R \to 0\).
- **Case** \(s^2 > 0\): Defining \(s > 0\), one discards as usual the divergent solution:
  \[
  F(R) \underset{R \to 0}{\sim} R^s \quad \longrightarrow \quad E_q = E_{\text{CoM}} + (s + 1 + 2q)\hbar\omega, \quad q \in \mathbb{N}
  \]
- **Case** \(s^2 < 0\): To make the Hamiltonian self-adjoint, one is forced to introduce an extra parameter \(\kappa\) (inverse of a
length, calculable via microscopic model). For $s = i|s|:
\begin{equation}
F(R) \sim \lim_{R \to 0} (\kappa R)^s - (\kappa R)^{-s}
\end{equation}

This breaks scaling invariance of the domain. In free space, a geometric spectrum of $N$-mers:
\begin{equation}
E_n \propto -\frac{\hbar^2 \kappa^2}{\bar{m}} e^{-2\pi n/|s|}, \quad n \in \mathbb{Z}
\end{equation}

For $N = 3$, this is the Efimov effect:

- **Efimov (1971):** Solution for three bosons ($1/a = 0$). There exists a single purely imaginary $s_3 \simeq i \times 1.00624$.

- **Efimov (1973):** Solution for three arbitrary particles ($1/a = 0$). Efimov trimers for two fermions (masse $m$, same spin state) and one impurity (masse $m'$) iff (Petrov, 2003)

  \begin{equation}
  \alpha \equiv \frac{m}{m'} > \alpha_c(2; 1) \simeq 13.6069
  \end{equation}
DOES THE UNITARY GAS EXIST?
MINLOS’S THEOREM (1995)
Theorem: In the $n+1$ fermionic problem, the Wigner-Bethe-Peierls Hamiltonian is self-adjoint and bounded from below iff

$$(n - 1) \frac{2\alpha(1 + 1/\alpha)^3}{\pi \sqrt{1 + 2\alpha}} \int_0^{\alpha \frac{\pi}{1+\alpha}} dt \ t \ \sin t < 1.$$ 

- $\alpha$ is mass ratio fermion/impurity
- Case $\alpha = 1$: No stable unitary gas for $n > 9$...
- Proof not included in Minlos’ paper. Nobody (not even Minlos) was able to reproduce the “missing proof”.
- Is is necessary? A physical test: look for occurrence of $s^2 < 0$ for $n = 3$: four-body Efimov effect!?
ARE THERE EFIMOVIAN TETRAMERS?

\[ E_n^{(4)} \propto -\frac{\hbar^2 \kappa_4^2}{m} e^{-2\pi n/|s_4|} \]

Negative results for bosons:

• Amado, Greenwood (1973): “There is No Efimov effect for Four or More Particles”. Explanation: Case of bosons, there exist trimers, tetramers decay.

• Hammer, Platter (2007), von Stecher, D’Incao, Greene (2009), Deltuva (2010): The four-boson problem (here \(1/a = 0\)) depends only on \(\kappa_3\), no \(\kappa_4\) to add.

• Key point: \(N = 3\) Efimov effect breaks separability in hyperspherical coordinates for \(N = 4\).

Here, we are dealing with fermions.
OUR DEFINITION OF N-BODY EFIMOV EFFECT

• To find $N$-body Efimov effect, one simply needs to calculate the exponents $s_N$, that is to solve the Wigner-Bethe-Peierls model at zero energy:

$$\psi_{E=0}(\vec{r}_1, \ldots, \vec{r}_N) = R^{s_N-(3N-5)/2} \phi(\vec{\Omega})$$

• The $N$-body Efimov effect takes place iff one of the $s_N^2$ is $< 0$.

• This statement makes sense if $\Delta_{\vec{\Omega}}$ self-adjoint for the Wigner-Bethe-Peierls contact conditions: There should be no $n$-body Efimov effect $\forall n \leq N - 1$. 

THE 3 + 1 FERMIONIC PROBLEM
(Castin, Mora, Pricoupenko, 2010)

• Three fermions (mass $m$, same spin state) and one impurity (mass $m'$)

• Our def. of 4-body Efimov effect requires a mass ratio

$$\alpha \equiv \frac{m}{m'} < \alpha_c(2; 1) \simeq 13.6069$$

• Calculate $E = 0$ solution in momentum space. An integral equation for Fourier transform of $A_{ij}$:

$$0 = \left[ \frac{1 + 2\alpha}{(1 + \alpha)^2} (k_1^2 + k_2^2) + \frac{2\alpha}{(1 + \alpha)^2} \vec{k}_1 \cdot \vec{k}_2 \right]^{1/2} D(\vec{k}_1, \vec{k}_2)$$

$$+ \int \frac{d^3k_3}{2\pi^2} \frac{D(\vec{k}_1, \vec{k}_3) + D(\vec{k}_3, \vec{k}_2)}{k_1^2 + k_2^2 + k_3^2 + \frac{2\alpha}{1+\alpha}(\vec{k}_1 \cdot \vec{k}_2 + \vec{k}_1 \cdot \vec{k}_3 + \vec{k}_2 \cdot \vec{k}_3)}$$

• $D$ has to obey fermionic symmetry.
RESULTS

• Four-body Efimov effect obtained for a single $s_4$, in channel $l = 1$ with even parity. Corresponding ansatz:

$$D(\vec{k}_1, \vec{k}_2) = \vec{e}_z \cdot \frac{\vec{k}_1 \times \vec{k}_2}{||\vec{k}_1 \times \vec{k}_2||} (k_1^2 + k_2^2)^{-\frac{(s_4 + 7/2)}{2}} F(k_2/k_1, \theta)$$

in the interval of mass ratio

$$\alpha_c(3; 1) \simeq 13.384 < \alpha < \alpha_c(2; 1) \simeq 13.607$$

• Strong disagreement with Minlos’ critical mass ratio for $n = 3$, $\alpha_c^{\text{Minlos}} \simeq 5.29$

• In experiments: Use optical lattice to tune effective mass of $^{40}\text{K}$ and $^{3}\text{He}^*$ away from $\alpha \simeq 13.25$
NUMERICAL VALUES OF $s_4 \in i\mathbb{R}$
CONCLUSION ON SYMMETRIES OF THE UNITARY GAS

• Unitary gas = gas of particles with interactions of infinite \( s \)-wave scattering length and negligible (true or effective) range

• Described by Wigner-Bether-Peierls zero-range model: Free Hamiltonian plus contact conditions

• Several physical properties result from scaling invariance of the model: E.g. undamped breathing mode of frequency \( 2\omega \) in an isotropic harmonic trap \( \rightarrow \) vanishing of bulk viscosity.

• Existence of unitary gas (even for fermions) not evident; may be destroyed by generalized \( N \)-body Efimov effect.

• In the \( n + 1 \) fermionic problem, sequence of critical mass ratios:

\[
\alpha_c(2; 1) = 13.6069 \ldots \quad \alpha_c(3; 1) = 13.384 \ldots \quad \alpha_c(4; 1) = ?
\]
Our publications on the subject


